Modeling Income Inequality: Exploring Transitions Between Quintiles Using Markov Chains

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Abstract

This project aims to create a model that will mathematically describe different characteristics of how an individual can transition between income brackets. Each income bracket, or quantile, makes up an equal percentage of the population with quantiles 1 through n being lowest income to highest income. By using Markov chains, we can investigate the probability of an individual moving from one quantile to the next. From this matrix, we can visualize different probabilities to describe movement throughout the quantiles and the necessary conditions to ensure an individual can move from the lowest to the highest quantile as quickly as possible.

Introduction

We begin with a model that does not consider retirement. This can be thought of as a base case, where p_i is the probability of an individual moving from i to $i+1$ and q_i is the probability of an individual moving from i to $i-1$. Since each row and column sums to 1, we can conclude that $q_{i+1} = p_i$ and adjust the matrix as such.

From here, we discuss mobility. The vector $\vec{u}$ describes the expected time that an individual arrives at the last state. If an individual is already in the last quantile, it will take them 0 years. Then, we can redefine $\vec{u}$ without the last component, and call it $\vec{h}$. We then use the P matrix to observe the expected times to get to the last quantile, which is referred to as the hitting time.

Matrices

$$P = \begin{pmatrix} 1-p_1 & p_1 & 0 & 0 & \cdots & \cdots & 0 \\ p_1 & 1-p_2-p_1 & p_2 & 0 & \ddots & & 0 \\ 0 & p_2 & 1-p_3-p_2 & p_3 & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & p_{n-2} & 1-p_{n-1}-p_{n-2} & p_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & p_{n-1} & 1-p_{n-1} \end{pmatrix}$$

$$\vec{u} = \begin{pmatrix} E(\tau_n|x_0=1) \\ E(\tau_n|x_0=2) \\ E(\tau_n|x_0=3) \\ \vdots \\ E(\tau_n|x_0=n) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix}$$

Plots of Hitting Times

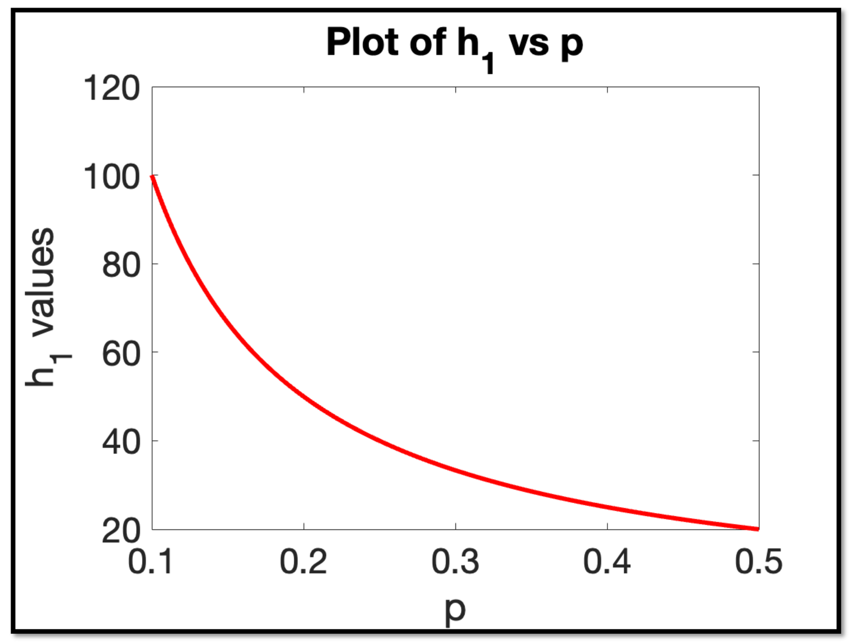

Figure 1. Plot of hitting time for p_1 with uniform distribution of p_i, where p_i is between 0.1 and 0.5.

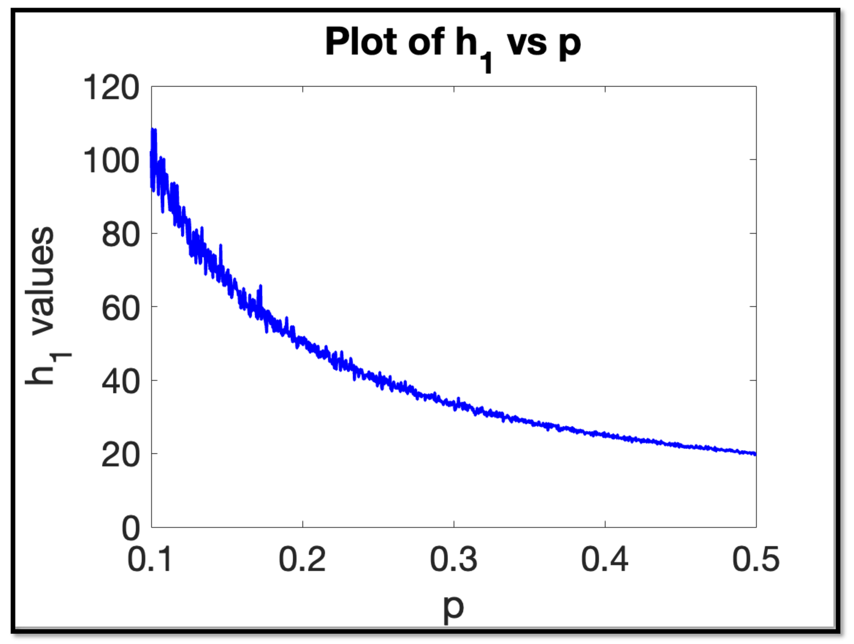

Figure 2. Plot of hitting time for p_1 with random β-distribution of p_i, where p_i is between 0.1 and 0.5.

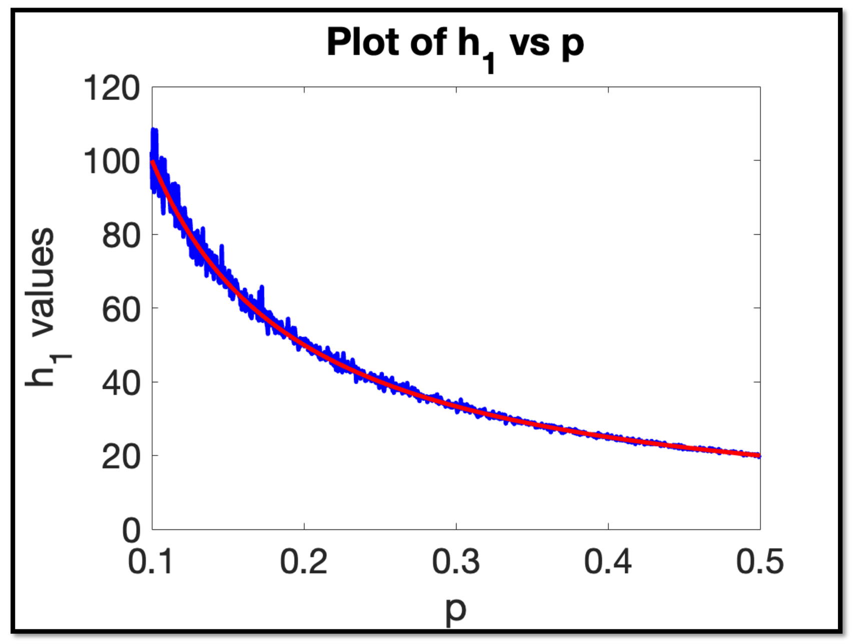

Figure 3. Plot of hitting time for p_1 with uniform distribution of p_i and p_1 with random β-distribution of p_i, where p_i is between 0.1 and 0.5.

Examining Probabilities of Hitting Times

Given $\vec{h}$ and the transition matrix $\tilde{P}$ as the $(n-1)\times(n-1)$ matrix of P, then we have that $(I-\tilde{P})\vec{h} = \vec{1}$.

We use $(n-1)$ because the last state does not have mobility.

From here we can solve for $\vec{h}$, and we get $h_i = \sum_{j=i}^{n-1} \frac{j}{p_j}$.

Furthermore, $h_1 = \frac{1}{p_1} + \frac{2}{p_2} + \cdots + \frac{n-1}{p_{n-1}}$.

Hypothesis

Let h_1 be the first component of $\vec{h}$ and $\widetilde{h_1}$ be a rearrangement of at least two p_i's. If $p_i < p_{i+1}$ for all i, then h_1 is the minimum of all possible $\widetilde{h_1}$.

Proof. Without loss of generality, we can rearrange p_j and p_{j+m} where $1 \le j \le n-1$ and $j+m \le n-1$.

$$\widetilde{h_1} = \frac{1}{p_1} + \frac{2}{p_2} + \cdots + \frac{j}{p_{j+m}} + \cdots + \frac{j+m}{p_j} + \cdots + \frac{n-1}{p_{n-1}}$$

Then we have

$$\widetilde{h_1} - h_1 = \frac{j}{p_{j+m}} + \frac{j+m}{p_j} - \frac{j}{p_j} - \frac{j+m}{p_{j+m}} = \frac{m}{p_j} - \frac{m}{p_{j+m}}$$

Since we have that $p_j < p_{j+m}$, then we have that $\frac{m}{p_j} - \frac{m}{p_{j+m}} > 0$ and thus $h_1 < \widetilde{h_1}$.

Discussion

Given a set of probabilities that can be calculated from a given data set, we can calculate the expected hitting time for an individual to move from the first quintile to the last. If the goal is to find or minimize a hitting time, we can establish conditions on the set of probabilities to find a smaller hitting time.

From *Income Inequality Matters, but Mobility is Just as Important*, by Carrol and Chen we can see that their transition matrix is defined as

$$\begin{pmatrix} 0.64 & 0.24 & 0.08 & 0.03 & 0.01 \\ 0.23 & 0.45 & 0.24 & 0.07 & 0.02 \\ 0.08 & 0.20 & 0.46 & 0.23 & 0.04 \\ 0.04 & 0.07 & 0.19 & 0.54 & 0.18 \\ 0.03 & 0.04 & 0.06 & 0.16 & 0.72 \end{pmatrix}$$

Then we can establish that an analogous P matrix would be

$$P = \begin{pmatrix} 0.64 & 0.36 & 0 & 0 & 0 \\ 0.36 & 0.45 & 0.33 & 0 & 0 \\ 0 & 0.33 & 0.46 & 0.27 & 0 \\ 0 & 0 & 0.27 & 0.54 & 0.18 \\ 0 & 0 & 0 & 0.18 & 0.72 \end{pmatrix}$$

Then the calculated hitting time h_1 would be

$$h_1 = \frac{1}{0.36} + \frac{2}{0.33} + \frac{3}{0.27} + \frac{4}{0.18} = 42.171717\ldots$$

If however, we arrange the p_i's to be increasing such as

$$\tilde{P} = \begin{pmatrix} 1-0.18 & 0.18 & 0 & 0 & 0 \\ 0.18 & 1-0.18-0.27 & 0.27 & 0 & 0 \\ 0 & 0.27 & 1-0.27-0.33 & 0.33 & 0 \\ 0 & 0 & 0.33 & 1-0.33-0.36 & 0.36 \\ 0 & 0 & 0 & 0.36 & 1-0.36 \end{pmatrix}$$

Then the new calculated hitting time $\widetilde{h_1}$ would be

$$\widetilde{h_1} = \frac{1}{0.18} + \frac{2}{0.27} + \frac{3}{0.33} + \frac{4}{0.36} = 33.162983\ldots$$

which is smaller than h_1, thus the system depicted by P is not optimal for mobility. Thus the system depicted by Carrol and Chen is not optimal for mobility.

Acknowledgements

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