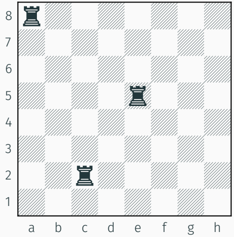
The Monoid of Non-attacking Rooks



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Joint Work with Mahir Bilen Can, Hayden Houser, and Diego Villamizar Rubiano

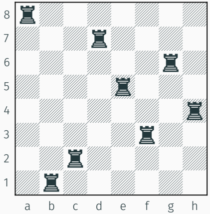
Non-Attacking Rooks

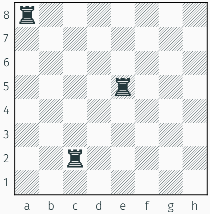
Non-Attacking Rooks

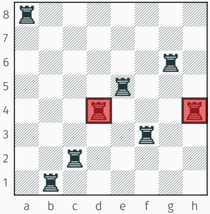
Definitions

- A **rook** is a chess piece that may move any number of spaces either horizontally or vertically per move.
- Two rooks are **non-attacking** if they are not be placed on the same row or column of the board.

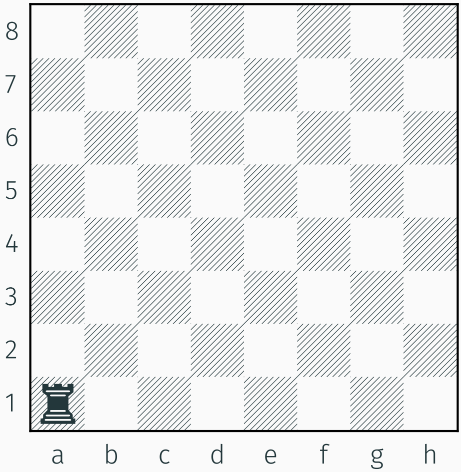
Example







Maximum Number of Non-attacking Rooks



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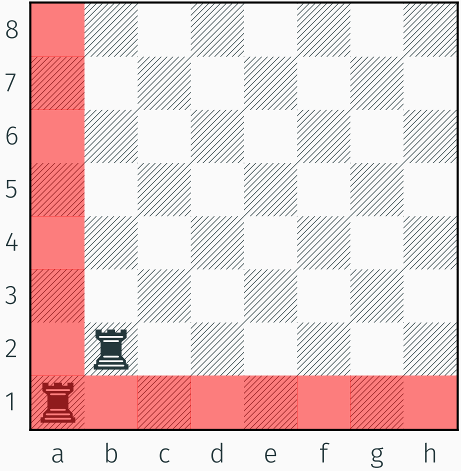

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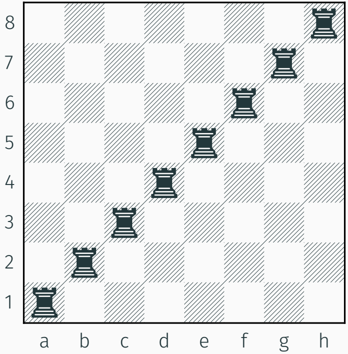
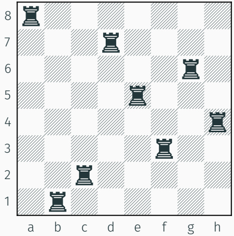


- The maximum number of non-attacking rooks that may be placed on an $n \times n$ chessboard is n.
- The number of ways of placing n non-attacking rooks on an $n \times n$ chessboard is $n! = n(n-1)(n-2)\cdots(2)(1)$.

Rook Monoid

Let M_n denote the algebra of $n \times n$ matrices over $\mathbb{C}$. Boards are in one-to-one correspondence with zero-one matrices with at most one nonzer0 entry in each row and column.



$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rook Monoid

The **Rook monoid** R_n consists of all zero-one matrices with at most one nonzero entry in each row and column.

Remark

Elements with n entries are in one-to-one correspondence with permutations of $[n]$.

Example

$$\mathsf{R}_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

The Symplectic Monoid MSp_n with $n = 2l$

The symplectic group is

$$\mathsf{Sp}_n = \left\{A \in \mathsf{GL}_n \mid A^\top J A = J\right\}$$

and the symplectic monoid is

$$\mathsf{MSp}_n = \left\{A \in \mathsf{M}_n \mid A^\top J A = A J A^\top = cJ \text{ for some } c \in \mathbb{C}\right\}.$$

Admissible Sets

Define an involution denoted by θ on the set $\mathsf{n} = 1, 2, \ldots, n$ such that

$$\theta(i) = n + 1 - i.$$

A subset I of n is considered **admissible** if $I \cap \theta(I) = \emptyset$.

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Example

If $n = 4$, then the admissible subsets of $\mathbf{n}$ are

$$\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}, \{1,2,3,4\}.$$

Let $I(A)$ and $J(A)$ represent the sets of indices for the nonzero columns and rows of $A = (a_{ji}) \in \mathsf{R}_n$, respectively.

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Theorem

The symplectic rook monoid is

$$\begin{aligned}\mathsf{RSp}_n &= \{A \in \mathsf{R}_n \mid A \textit{ is singular and } I(A) \textit{ and } J(A) \textit{ are admissible } \} \cup W \\ &\simeq \left\{A \in \mathsf{R}_n \mid AJA^\top = A^\top JA = 0 \textit{ or } J\right\}.\end{aligned}$$

Rook Poset

For $x = (x_{ij}) \in \mathrm{R}_n$ define the sequence $(x_1, \ldots, x_n)$ by

$$x_j = \begin{cases} 0 & \text{if the } j \text{ -th column consists of zeros,} \\ i & \text{if } x_{ij} = 1 \end{cases}$$

Let x denote both the n-tuple and the matrix.

Example

$$x = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = (2, 3, 0).$$

Rook Poset Order

Let $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_k\}$ be two sets of integers such that $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_k$. Then define a partial order as follows

$$\{i_1, \ldots, i_k\} \leqslant \{j_1, \ldots, j_k\} \iff i_1 \leq j_1, i_2 \leq j_2, \ldots, i_k \leq j_k$$

Let $x = (x_1, \ldots, x_n) \in \mathrm{R}_n$. For $i \in \{1, \ldots, n\}$, define

$$\tilde{x}(i) := \{x_1, \ldots, x_i\}$$

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Theorem

Let $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ be two elements in R_n. Then $x \leq y$ if and only if for every $i \in \{1, \ldots, n-1\}$, we have $\tilde{x}(i) \leq \tilde{y}(i)$.

Corollary

Let $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ be two elements in R_{Sp_n}. Then $x \leq y$ if and only if for every $i \in \{1, \ldots, n-1\}$, we have $\tilde{x}(i) \leq \tilde{y}(i)$.

Example

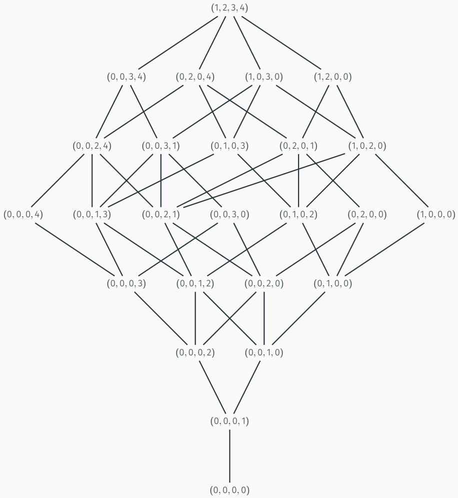
Let $x = (3, 1, 5, 2, 4)$ and $y = (5, 2, 4, 3, 1)$ be elements of R_5. Then,

$$\begin{aligned} \tilde{x}(1) = \{3\} &\quad\leq\quad \{5\} = \tilde{y}(1) \\ \tilde{x}(2) = \{1, 3\} &\quad\leq\quad \{2, 5\} = \tilde{y}(2) \\ \tilde{x}(3) = \{1, 3, 5\} &\quad\leq\quad \{2, 4, 5\} = \tilde{y}(3) \\ \tilde{x}(4) = \{1, 2, 3, 5\} &\quad\leq\quad \{2, 3, 4, 5\} = \tilde{y}(4), \end{aligned}$$

Thus, $x \leq y$.

Let B_n be the upper triangular matrices. Then,

$$\mathsf{BSp}_n = \mathsf{B}_n \cap \mathsf{RSp}_n := \{x \in \mathsf{RSp}_n : x \leq 1\}.$$



Count on the Upper Triangular Submonoid

Arc-Diagram

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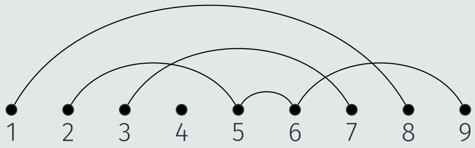
An Arc-Diagram on 9 Vertices



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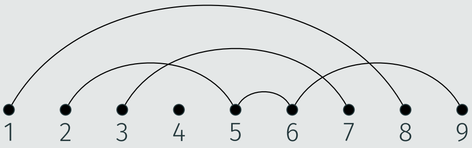

An arc-diagram's subchains represent the blocks of the corresponding set partition.

The number of set partitions of $S = 1, ..., n$ into k blocks, denoted by $S(n, k)$, and called the (n, k)**-th Stirling number of the second kind**, is given by the formula

$$S(n,k) = \frac{1}{k!}\sum_{i=1}^{k}(-1)^i\binom{k}{i}(k-i)^n.$$

The recurrence formula for the Stirling numbers of the second kind is well-known:

$$S(l+1,k) = S(l,k-1) + kS(l,k)$$

where

$$S(l,k) = \begin{cases} 1 & \text{if } l = k = 0 \\ 0 & \text{if } l > 0 \text{ and } k = 0 \\ 0 & \text{if } l < 0 \text{ or } k < 0 \text{ or } l < k \end{cases}$$

Count on B_n

Denote the subsemigroup of nilpotent elements in B_n by B_n^{nil}. Then define a bijection $\mathrm{B}_n \longrightarrow \mathrm{B}_{n+1}^{nil}$ by

$$A \longmapsto \tilde{A} := \begin{bmatrix} 0 & & \\ \vdots & A & \\ 0 & \dots & 0 \end{bmatrix} \in \mathrm{B}_{n+1} \quad (A \in \mathrm{B}_n)$$

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Number of elements of B_n of rank $n+1-k$

Therefore, for $k \in \{1, \dots, n+1\}$,

$$S(n+1, k) = |\{A \in \mathsf{B}_n : \operatorname{rank} A = n+1-k\}|$$

Count on B_n

The number of elements of B_n is given by the summation

$$b_{n+1} := \sum_{k=0}^{n+1} S(n+1, k),$$

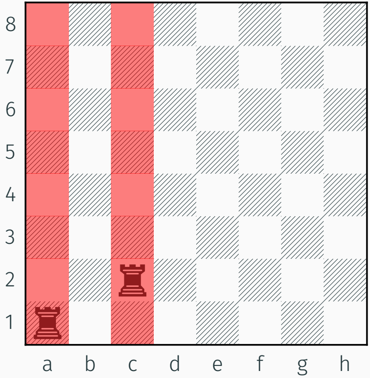
which is called the $(n+1)$th Bell number.

Number of Elements of R_n of rank k

The number of elements of R_n of rank k is given by

$$|\{A \in \mathsf{R}_n : \mathrm{rank}(A) = k\}| = \binom{n}{k} \frac{n!}{(n-k)!}.$$

$$|\{A \in \mathsf{R}_8 : \mathrm{rank}(A) = 2\}| = \binom{8}{2}\frac{8!}{(8-2)!} = \binom{8}{2} 8 * 7.$$



Every element A of R_n has a triangular decomposition in R_n,

$$A = A_l + A_d + A_u$$

where A_l is a strictly lower triangular matrix, A_d is a diagonal matrix, and A_u is a strictly upper triangular matrix.

Proposition 3

Let $S_{a,b,c}(n)$ denote the number of elements $A \in \mathsf{R}_n$ such that $\mathrm{rank}(A_l) = a$, $\mathrm{rank}(A_d) = b$, and $\mathrm{rank}(A_u) = c$. Then we have

$$\begin{aligned}\binom{n}{k}\frac{n!}{(n-k)!} &= \sum_{a+b+c=k} S_{a,b,c}(n) \\ &= \sum_{a+b+c=k} \binom{n}{b} S(n+1, n+1-a) S(n+1, n+1-c)\end{aligned}$$

The Folding Operators

Consider the folding operators on an element of R_{Sp_8}

Folding from Top to Bottom, F_{TB}

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Folding Operators

Consider the folding operators on an element of R_{Sp_8}

Folding from Left to Right, F_{LR}

$$
\left[\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\longrightarrow
\begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

F_{TB} and F_{LR} can be composed. In fact, $F_{TB}F_{LR} = F_{LR}F_{TB}$

The Folding Map, F

Let $F : \mathsf{R}_{Sp_n} \to \mathsf{R}_l$ defined by the composition of folding operators.

Proposition 4

The folding map is a surjective map from R_{Sp_n} onto the rook monoid R_l. The restricted folding map, $F' := F|_{\mathbf{BSp}_n}$, is also surjective.

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The Folding Map, F

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The folding map is a surjective map from R_{Sp_n} onto the rook monoid R_l. The restricted folding map, $F' := F|_{\mathbf{BSp}_n}$, is also surjective.

We're almost there!

We are now ready to count the number of elements of $\mathbf{BSp}_n$ by "unfolding" the elements of R_l first horizontally from bottom to top, and then vertically from right to left.

The Preimage of J_2 under $F' : \mathbf{B}_{Sp_4} \to \mathbf{R}_2$

Let's compute the preimage of $J_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ under $F' : \mathbf{B}_{Sp_4} \to \mathbf{R}_2$ or equivalently, determine the set $F_{LR}^{-1}\left(F_{TB}^{-1}(J_2)\right) \cap \mathbf{B}_{Sp_4}$.

$F_{TB}^{-1}(J_2)$

$$\begin{tikzcd}
\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ \hline 0 & 1 \\ 0 & 0 \end{array}\right] \arrow[dr, "F_{TB}"] & & \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right] \arrow[dl, "F_{TB}"'] \\
& \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} &
\end{tikzcd}$$

Since we are looking for the upper triangular elements in the preimage, the lower halves of the 4×2 matrices must be upper triangular.

$F_{LR}^{-1}\left(F_{TB}^{-1}(J_2)\right) \cap \mathsf{B}_{Sp_4}$

$$\left[\begin{array}{cc:cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{F_{LR}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc:cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{F_{LR}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We find that, in total, there are four matrices that fold onto J_2.

Using this technique, we find that the number of elements of BSp_n that lie in preimage of the folding map F'.

Theorem 5

The number of elements of rank k in BSp_n is given by

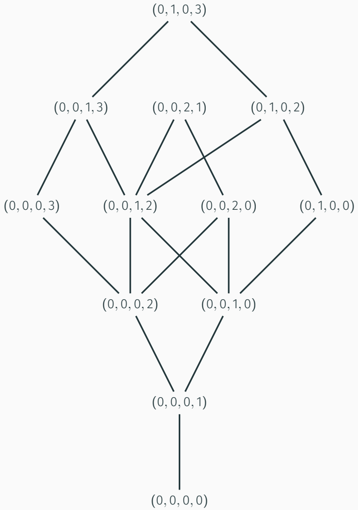
$$\sum_{a+b+c=k} 2^{a+c}3^b \binom{l}{b} S(l+1, l+1-a)S(l+1, l+1-c)$$

where $(a, b, c) \in \mathbb{Z}^3_{\geq 0}$.

Stirling Posets

The Hasse diagram of $\overline{\mathrm{BSP}_n}^{nil}$

The subvariety $\overline{B}^{nil} := \{x \in \overline{B} : \; x^m = 0 \text{ for some } m \in \mathbb{Z}_+\}$ will be called the *nilpotent semigroup* of $\overline{B}$.



Stirling Posets

Let A and B be two arc-diagrams on n vertices. Then B is said to cover A, and denoted by $A \prec B$, if it is obtained from A by one of the following three operations:

1. **The shortening of an arc of A.**
   With this operation, move exactly one endpoint of an arc to another vertex so that the resulting arc is shortened as minimally as possible but the number of crossings does not change.

2. **Deleting a crossing.**
3. **Adding a new arc.**

Stirling Posets of Type A

Let A and B be two arc-diagrams on n vertices. Then B is said to cover A, and denoted by $A \prec B$, if it is obtained from A by one of the following three operations:

1. **The shortening of an arc of A.**
2. **Deleting a crossing.**
   With this operation, interchange the rightmost endpoints of two crossing arcs so that they become a pair of non-crossing and nested arcs; require in this operation that only one crossing is deleted as a result of this operation.

3. **Adding a new arc.**

Stirling Posets of Type A

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1. **The shortening of an arc of A.**
2. **Deleting a crossing.**
3. **Adding a new arc.**
   With this operation, a new arc is introduced between two vertices in such a way that the new arc is not under any other (older) arcs and the endpoints of the new arc are as far from each other as possible.

Let A and B be two arc-diagrams on n vertices. Then B is said to cover A, and denoted by $A \prec B$, if it is obtained from A by one of the following three operations:

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3. **Adding a new arc.**

Theorem

The arc-diagram poset $(\mathcal{A}_n, \preceq)$ is isomorphic to $\left(B_n^{\text{nil}}, \leq\right)$, hence it is a bounded, graded, and EL-shellable poset.

Definition

An arc diagram $A \in \mathcal{A}_n$ is **symplectic** if $I \cap \theta(I) = \emptyset$ and $J \cap \theta(J) = \emptyset$ where θ is the involution defined by $\theta(l) = n - l + 1$. We denote the set of all symplectic arc diagrams as $\mathcal{A}Sp_n$.

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Suppose A is a symplectic Stirling arc diagram. Then

1. each vertex of A can only be the start of one arc
2. each vertex of A can only be the end of one arc
3. if a vertex of v of A is the start of an arc and the end of an arc, the vertex $\theta(v)$ is **empty**, i.e. $\theta(v)$ is not the start or end of an arc in A.

Symplectic Arc Diagrams

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Lemma

The set of nilpotent symplectic rooks, $\overline{BSp}_n^{nil}$, is bijective to the set of symplectic Stirling arc diagrams, $\mathcal{A}Sp_n$.

Consider the symplectic rook

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which may be written as $x = (0, 1, 0, 2)$. To construct the arc diagram, we form the arcs $\{1, 2\}$ and $\{2, 4\}$ on a chain of four vertices.

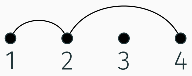

Figure 1: A symplectic arc diagram corresponding to $\pi = 124|3$

Symplectic Stirling Posets

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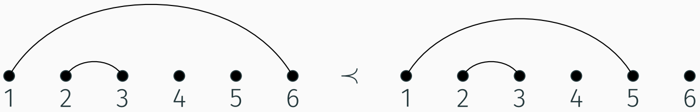

2. The shortening of two repeated arcs of A
3. Deleting a crossing.
4. Adding a new arc.

Symplectic Stirling Posets

Let A and B be two arc-diagrams on n vertices. Then B is said to cover A, and denoted by $A \prec B$, if it is obtained from A by one of the following three operations:

1. **The shortening of an arc of A.**
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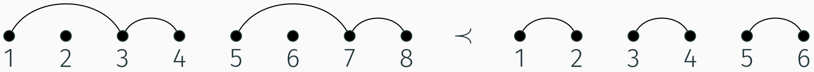

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Theorem

The symplectic Stirling poset $(\mathcal{A}Sp_n, \preceq)$ is isomorphic to $\left(\overline{BSp_n}^{\mathrm{nil}}, \leq\right)$.

$(\mathcal{A}Sp_n, \preceq) \cong \left(\overline{BSp}_n^{\mathsf{NIL}}, \leq\right)$

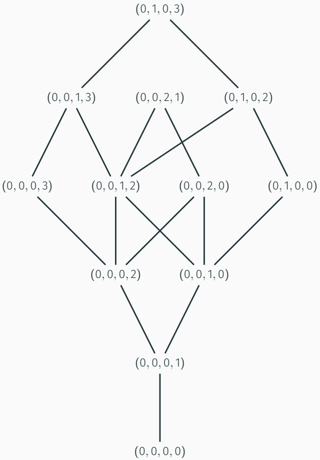

Figure 2: $\left(\overline{BSp}_4^{\mathrm{nil}}, \leq\right)$

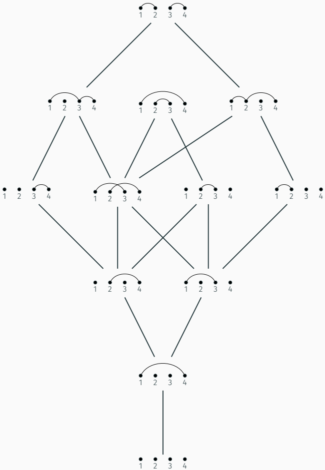

Figure 3: $(\mathcal{A}Sp_4, \preceq)$

Denote the set of strictly upper triangular symplectic rooks as RSp_n and let $\mathsf{RSp}_{n,k}$ be the set of strictly upper triangular symplectic rooks of rank k.

Lemma

The following are true

1. $|RSp_{n,k}| = 0$, *for all* $k > \ell$.
2. $|RSp_{n,0}| = 1$.
3. $|RSp_{n,1}| = \binom{n}{2}$.
4. $|RSp_{2\ell,2}| = \frac{\binom{2\ell}{2}(2\ell^2-5\ell+4)-\ell}{2}$

Consider two sets of n symbols $[n] = 1, 2, ..., n$ and $[\overline{n}] = \overline{1}, \overline{2}, \ldots, \overline{n}$, with $i \neq \bar{j}$, for any i, j. Define $X_n = [n] \cup [\overline{n}]$.

Definition

Permutations of X_n such that $\sigma(\bar{j}) = \overline{\sigma(j)}$ for all $j \in X_n$ are **signed permutations** or **permutations of type B**.

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Example

Consider the following signed permutation.

$$\sigma = \begin{pmatrix} 1\,2\,3\,4\,\bar{1}\,\bar{2}\,\bar{3}\,\bar{4} \\ \bar{2}\,1\,3\,\bar{4}\,2\,\bar{1}\,\bar{3}\,4 \end{pmatrix}.$$

Since $\sigma(\bar{j}) = \overline{\sigma(j)}$, we may condense this notation and express σ in line notation as

$$\sigma = \bar{2}13\bar{4}.$$

Symplectic Rooks with ℓ Arcs

A **type B derangement** is a type B permutation without fixed points. The number of derangements of type B of length n is given by d_n^B.

Theorem

For a positive integer ℓ, we have that

$$\left|RSp_{2\ell,\ell}\right| = d_\ell^B.$$

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Bijection:

Consider the following function

$$\varphi : \mathcal{D}_\ell^B \longrightarrow \mathrm{RSp}_{2\ell,\ell},$$

given by $\varphi(\pi) = \{f(i, \pi(i)) : i \in [\ell]\}$ and where f is defined as follows

$$f(i,j) = \begin{cases} (i, \theta(|j|)) & \text{if } j < 0 \\ (i,j) & 0 < i < j \\ (\theta(i), \theta(j)) & \text{if } 0 < j < i. \end{cases}$$

To illustrate the function φ, consider the element $3\bar{2}1 \in \mathcal{D}_\ell^B$. Then,

$$f(1,3) = (1,3) \text{ since } 0 < 1 < 3$$
$$f(2,\bar{2}) = (2,\theta(2)) = (2,5) \text{ since } \bar{2} < 0$$
$$f(3,1) = (\theta(3),\theta(1)) = (4,6) \text{ since } 1 > 0 \text{ and } 3 > 1$$

The symplectic arc-diagram is then defined by the arcs $\{1,3\}$, $\{2,5\}$, and $\{4,6\}$ shown in Figure 4.

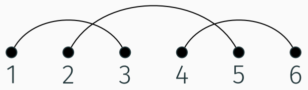


Figure 4: $\varphi(3\bar{2}1)$

$(\mathcal{A}Sp_n, \preceq) \cong \left(\overline{BSp}_n^{\mathsf{NIL}}, \leq\right)$

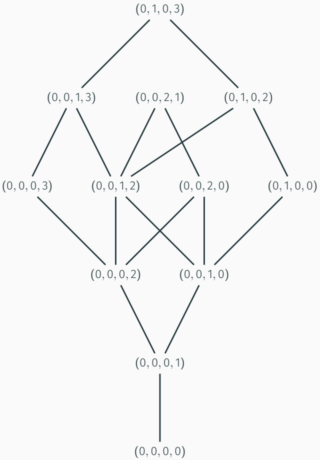


Figure 5: $\left(\overline{BSp}_4^{\mathrm{nil}}, \leq\right)$

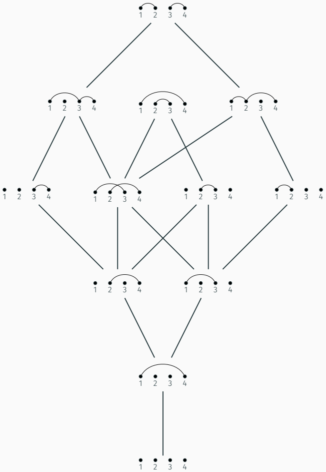


Figure 6: $(\mathcal{A}Sp_4, \preceq)$

Maximal Symplectic Rooks

Denote the set of maximal symplectic arc diagrams as $\mathsf{MSp}_{2\ell}$.

$n = 2l$	$\lvert\overline{BSp}_n^{nil}\rvert$	$\lvert\mathcal{MR}Sp_n\rvert$
2	2	1
4	12	2
6	96	5
8	1008	12
10	12960	32
12	196800	88
14	3440640	247
16	-	712
18	-	2084

Table 1: Number of Maximal Symplectic arc diagrams

Let $I(A)$ be the set of end points and $J(A)$ be the set of start points of the arcs of A.

Definition

A symplectic arc diagram is θ-**symmetric** if

1. if $i \in I(A)$ and $i \notin J(A)$, then $\theta(i) \in J(A)$
2. if $j \in J(A)$ and $j \notin I(A)$, then $\theta(j) \in I(A)$
3. if $i \in I(A)$ and $i \in J(A)$, then $\theta(i) \notin I(A)$ and $\theta(i) \notin I(A)$
4. if $i \notin I(A)$ and $i \notin J(A)$, then $\theta(i) \in I(A)$ and $\theta(i) \in I(A)$

All maximal symplectic arc diagrams on $n = 2\ell$ vertices

1. have ℓ arcs and
2. have no crossings,
3. is θ-symmetric

Consider $x = (0, 1, 2, 3, 0, 0, 0, 5)$. Then $A(x) \in \mathcal{A}Sp_8$ and

$$I(A) = \{2, 3, 4, 8\}$$
$$J(A) = \{1, 2, 3, 5\}$$

and thus, $A(x)$ is θ-symmetric and pictured below.

Figure 7: θ-symmetry

Proposition

A symplectic arc diagram, A, on $n = 2\ell$ vertices is maximal in $(\mathcal{A}Sp_n, \prec)$ if and only if A has ℓ arcs, is noncrossing, and is θ-symmetric,

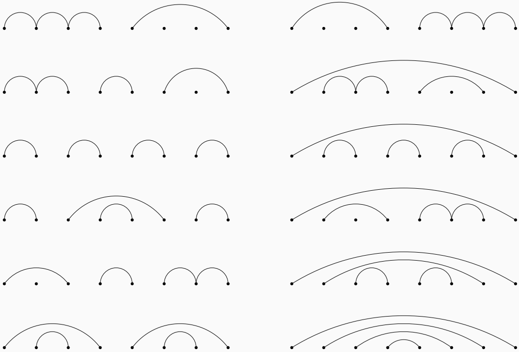


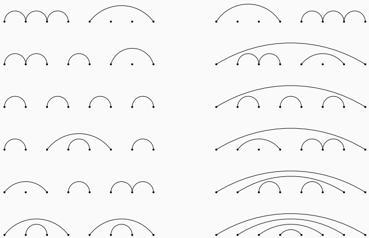
Figure 8: Elements of $\mathcal{MR}Sp_8$

Theorem

The number of maximal symplectic arc diagrams of length $n = 2\ell$ is given by

$$|\mathcal{MRS}p_n| = \sum_{i=0}^{\ell} M(2i)S(n-2i)$$

such that $M(i)$ is number of maximal arcs on i and $S(n-2i)$ counts the number of possible L/R pairs.

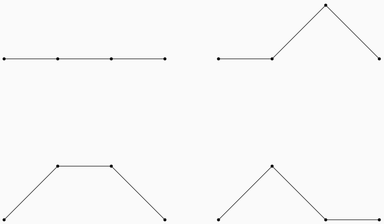


Definition

A **Motzkin path** P of size n is a lattice path in the integer plane $\times$ from $(0, 0)$ to $(n, 0)$ which never passes below the x-axis and whose permitted steps are the up step $u = (1, 1)$, the down step $d = (1, -1)$, and the horizontal step $h = (1, 0)$.

Example

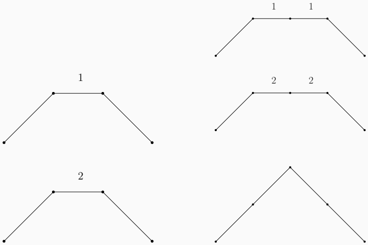
The elements of $\mathcal{M}_3$ can be expressed as Motzkin words of size, $\mathcal{M}_3 = \{uhd, udh, hud, hhh\}$ or as Motzkin paths.



Conjecture

Let $A \in \mathcal{MRS}p_n$. Partition $A = L|M|R$ as described in the previous proof. The set of L/R pairs, denoted $S(k)$ is in bijection with Motzkin paths of length $k-2$ with 2-colored horizontal steps determined by height.

Motzkin Paths from Left Components of Length 3 and 4 of $\mathcal{MRS}p_8$



Questions?