Number field analogue of Jacobi theta relation and zeros of Dedekind zeta function on $\mathrm{Re}(s) = 1/2$

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Outline

- Riemann zeta function $\zeta(s)$

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- Future directions

Riemann zeta function[1]:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{for} \quad \mathrm{Re}(s) > 1.$$

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$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

Bernhard Riemann (1826-1866)

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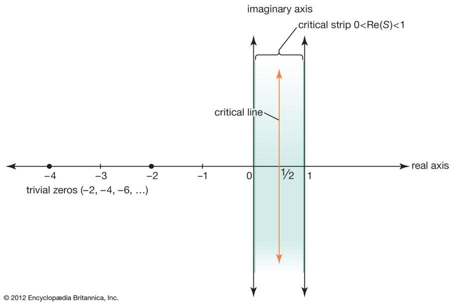
Riemann hypothesis and Hardy's result

■ Riemann conjectured that all the non-trivial zeros of $\zeta(s)$ lie on $\mathrm{Re}(s) = 1/2$ line.



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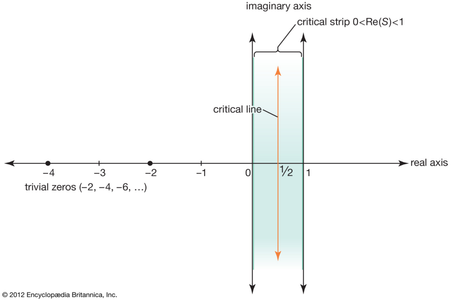

■ Hardy[2] proved that infinitely many non-trivial zeros of the Riemann zeta function lie on the critical line.

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♦ Let $\mathbb{F}$ be any number field with the degree of extension $[\mathbb{F} : \mathbb{Q}] = d$. Let $\mathcal{O}_{\mathbb{F}}$ be its ring of integers and $\mathfrak{N}$ be the norm map of $\mathbb{F}$ over $\mathbb{Q}$. Define $\mathtt{a}_{\mathbb{F}}(n) :=$ # ideals in $\mathcal{O}_{\mathbb{F}}$ with norm n.

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$$\zeta_{\mathbb{F}}(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_{\mathbb{F}}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s} = \sum_{n=1}^{\infty} \frac{\mathsf{a}_{\mathbb{F}}(n)}{n^s}$$

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♦ When $\mathbb{F} = \mathbb{Q}$ then $\zeta_{\mathbb{F}}(s) = \zeta(s)$

Functional equation for Dedekind zeta function

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Theorem (Dedekind)

The Dedekind zeta function has an analytic continuation in the whole complex plane except for a simple pole at $s = 1$. It also satisfies the following functional equation relating its values at s and $1 - s$:

$$\Omega_{\mathbb{F}}(s) = \Omega_{\mathbb{F}}(1 - s),$$

where $\Omega_{\mathbb{F}}(s) = \left(\frac{D}{4^{r_2}\pi^d}\right)^{\frac{s}{2}} \Gamma^{r_1}\left(\frac{s}{2}\right) \Gamma^{r_2}(s)\zeta_{\mathbb{F}}(s)$.

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It is known that $\zeta_{\mathbb{F}}(s)$ has a simple pole at $s = 1$ with the residue given by the class number formula:

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$$\lim_{s \to 0} \frac{\zeta_{\mathbb{F}}(s)}{s^r} = -\frac{R_{\mathbb{F}} h_{\mathbb{F}}}{w_{\mathbb{F}}} := C_{\mathbb{F}}. \quad (1)$$

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Note that $C_{\mathbb{F}}$ is negative for any number field.

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Definition

The Voronoi-Steen function $V(z; a_1, a_2, \ldots, a_n)$ is defined by, for $|\operatorname{Arg}(z)| < \frac{\pi n}{2}$,

$$V(z; a_1, a_2, \ldots, a_n) := \frac{1}{2\pi i} \int_{(C)} \prod_{j=1}^{n} \Gamma(s + a_j) z^{-s} ds,$$

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- $V(z; 0) = e^{-z}, \operatorname{Re}(z) > 0,$
- $V(z; a, b) = 2z^{\frac{1}{2}(a+b)} K_{a-b}(2z^{\frac{1}{2}}), |\operatorname{Arg}(z)| < \pi,$

where K_ν is the modified Bessel function of second kind.

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$$\tilde{Z}_{r_1,r_2}(x) = \frac{2}{2^{r_2}\pi^{\frac{r_2}{2}}} V\left(\frac{x^2}{4^{r_2}}\middle|\bar{0}_{r_1+r_2}, \left(\bar{\frac{1}{2}}\right)_{r_2}\right), \quad \text{for } |\operatorname{Arg}(x)| < \frac{\pi d}{4}.$$

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where $\bar{a}_m$ is the m-tuple whose all entries equal a.

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Recall that $\zeta(s)$ satisfies the following functional equation:

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Riemann proved the functional equation using the following Jacobi theta relation:

Jacobi theta relation

For $\mathrm{Re}(x) > 0$, we have

$$W_1\left(\frac{1}{x}\right) = \sqrt{x}W_1(x),$$

where $W_1(x)$ is Jacobi theta function, defined as

$$W_1(x) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x}.$$

[3] W. L. Ferrar, *Some Solutions of The Equation* $F(t) = F(t^{-1})$, J. London Math. Soc., 1**(2)**, 99–103, 1936.

Equivalence of Functional Equation for $\zeta^2(s)$

In 1936, Ferrar[3] gave a proof of the Ramanujan-Koshliakov formula using the functional equation for $\zeta^2(s)$.

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Ramanujan-Koshliakov Formula

For $\mathrm{Re}(x) > 0$,

$$W_2\left(\frac{1}{x}\right) = \sqrt{x}W_2(x).$$

Here, $W_2(x)$ is given by

$$W_2(x) = \gamma - \log(4\pi) + \log(\sqrt{x}) + 4\sum_{n=1}^{\infty} d(n)\mathrm{K}_0(2n\pi\sqrt{x}),$$

where γ is the Euler-Mascheroni constant.

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But one can prove the converse as well.

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Equivalence for functional equation for $\zeta^k(s)$

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Can we find a similar equivalence theta relation for the functional equation of the Dedekind zeta function $\zeta_{\mathbb{F}}(s)$?

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Question

Can we find a similar equivalence theta relation for the functional equation of the Dedekind zeta function $\zeta_{\mathbb{F}}(s)$? What about higher powers of $\zeta_{\mathbb{F}}(s)$?

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Dirichlet series

Some arithmetic functions

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Equivalence for functional equation of $\zeta_{\mathbb{F}}^{k}(s)$

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Theorem (Bansal, M.)

Let k be a positive integer and $\mathbb{F}$ be any number field of degree d. Then the functional equation for $\zeta_{\mathbb{F}}^k(s)$ given by

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is equivalent to the theta relation

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$$W_{\mathbb{F},k}(x) := \sum_{n=1}^{\infty} \mathsf{a}_{\mathbb{F},k}(n)\tilde{Z}_{kr_1,kr_2}\left(\frac{2^{kr_2}\pi^{kd/2}n\sqrt{x}}{\sqrt{D^k}}\right) - R_0(x), \quad \text{and}$$

$$R_0(x) = \frac{1}{(k-1)!}\lim_{s\to 0}\frac{\mathrm{d}^{k-1}}{\mathrm{d}s^{k-1}}\left(s^k\Omega_{\mathbb{F}}^k(s)x^{-s/2}\right).$$

Corollaries

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$$W_{\mathbb{F},1}(x) = \sum_{n=1}^{\infty} \mathsf{a}_{\mathbb{F}}(n)\tilde{Z}_{r_1,r_2}\left(\frac{2^{r_2}\pi^{d/2}n\sqrt{x}}{\sqrt{D}}\right) - 2^{r_1}C_{\mathbb{F}},$$

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When $\mathbb{F} = \mathbb{Q}$, it reduces to the Jacobi theta function,

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An interesting implication

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Corollary (2)

The functional equation for $\zeta(s)^k$ given by

$$\pi^{-ks}\Gamma\left(\frac{s}{2}\right)^k \zeta(s)^k = \pi^{-k(1-s)}\Gamma\left(\frac{1-s}{2}\right)^k \zeta(1-s)^k,$$

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Remark

Putting $k = 1$ and $k = 2$ in the last corollary gives the equivalences for functional equation of $\zeta(s)$ and $\zeta^2(s)$ with Jacobi theta relation and Ramanujan-Koshliakov formula respectively.

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For any $x > 0$, we have

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \exp\left(-\frac{x}{n^2}\right)$$
$$= \sqrt{\frac{\pi}{x}} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \exp\left(-\frac{\pi^2}{n^2 x}\right) - \frac{1}{2\sqrt{\pi}} \sum_{\rho} \left(\frac{\pi}{\sqrt{x}}\right)^{\rho} \frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)},$$

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where the sum over ρ runs through the non-trivial zeros of $\zeta(s)$.

Number field analogue of Hardy-Littlewood-Ramanujan identity

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Identity of Dixit, Gupta and Vatwani

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Let $\alpha, \beta \in \mathbb{C}$ be such that $\alpha\beta = \frac{4^{r_2}\pi^d}{D}$, then we have

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\frac{\mu_{\mathbb{F},1}(n)}{n}Z_{r_1,r_2}\left(\frac{\alpha}{n}\right) - \sqrt{\beta}\sum_{n=1}^{\infty}\frac{\mu_{\mathbb{F},1}(n)}{n}Z_{r_1,r_2}\left(\frac{\beta}{n}\right)$$
$$= \frac{1}{\sqrt{\alpha}}R_0(\alpha) - \frac{1}{\sqrt{\beta}}R_0(\beta) + \frac{1}{2}\left[\frac{1}{\sqrt{\alpha}}\sum_{\rho}R_\rho(\alpha) - \frac{1}{\sqrt{\beta}}\sum_{\rho}R_\rho(\beta)\right],$$

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where the terms $R_0(\alpha)$ and $R_\rho(\alpha)$ are given by

$$R_0(\alpha) = \frac{1}{(r-1)!}\lim_{s\to 0}\frac{\mathrm{d}^{r-1}}{\mathrm{d}s^{r-1}}\left(s^r\alpha^s\frac{\Gamma^{r_1}\left(\frac{1-s}{2}\right)\Gamma^{r_2}\left(1-s\right)}{\zeta_{\mathbb{F}}(s)}\right),$$
$$R_\rho\left(\alpha\right) = \alpha^\rho\frac{\Gamma^{r_1}\left(\frac{1-\rho}{2}\right)\Gamma^{r_2}\left(1-\rho\right)}{\zeta'_{\mathbb{F}}(\rho)}.$$

Equivalence between theta-type identities and functional equations of $\zeta^k(s)$ and $\zeta_{\mathbb{F}}^k(s)$, $k \in \mathbb{Z}^*$.

Theta-type relations	**Functional equation**	**Theta-type relations**
Jacobi theta relation	$\zeta(s) \Longleftrightarrow \zeta^{-1}(s)$	Hardy, Littlewood and Ramanujan identity
Ramanujan-Koshliakov identity	$\zeta^2(s) \Longleftrightarrow \zeta^{-2}(s)$	The case when $k = 2$ and $\mathbb{F} = \mathbb{Q}$
Ferrar's identity	$\zeta^k(s) \Longleftrightarrow \zeta^{-k}(s)$	A generalization of Hardy, Littlewood and Ramanujan (when $\mathbb{F} = \mathbb{Q}$)
Number field analogue of Jacobi theta relation	$\zeta_{\mathbb{F}}(s) \Longleftrightarrow \zeta_{\mathbb{F}}^{-1}(s)$	An identity of Dixit, Gupta and Vatwani
Number field analogue of Ferrar's identity	$\zeta_{\mathbb{F}}^k(s) \Longleftrightarrow \zeta_{\mathbb{F}}^{-k}(s)$	A generalization of Dixit, Gupta and Vatwani

Non-trivial zeros on $Re(s) = 1/2$

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• In 1968, Chandrasekharan and Narasimhan proved this result for the Dedekind zeta function associated to quadratic number fields.

• In 1970, Berndt proved that, if $r_1 \leq 3$ where $d = r_1 + 2r_2$, then the Dededkind zeta function has infinitely many non-trivial zeros on the critical line.

Analogue of Hardy's Result

Theorem (Bansal, M.)

For any number field $\mathbb{F}$, the Dedekind zeta function $\zeta_{\mathbb{F}}(s)$ has infinitely many non-trivial zeros on the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

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Outline of proof:

We first define the following entire function:

$$\begin{aligned}\xi_{\mathbb{F}}(s) &:= \frac{1}{2}s(s-1)\left(\frac{D}{4^{r_2}\pi^d}\right)^{\frac{s}{2}}\Gamma^{r_1}\left(\frac{s}{2}\right)\Gamma^{r_2}(s)\zeta_{\mathbb{F}}(s)\\ &= \frac{1}{2}s(s-1)\Omega_{\mathbb{F}}(s) = \xi_{\mathbb{F}}(1-s). \qquad (2)\end{aligned}$$

Putting $s = \frac{1}{2} + it$ in the above equation and denoting it as $\Xi_{\mathbb{F}}(t)$, we get

Proof contd...

$$\xi_{\mathbb{F}}\left(\frac{1}{2}+it\right)=\Xi_{\mathbb{F}}(t):=\left(-\frac{1}{8}-\frac{t^2}{2}\right)\left(\frac{D}{4^{r_2}\pi^d}\right)^{\frac{1}{4}+\frac{it}{2}}\Gamma^{r_1}\left(\frac{1}{4}+\frac{it}{2}\right)$$
$$\times\,\Gamma^{r_2}\left(\frac{1}{2}+it\right)\zeta_{\mathbb{F}}\left(\frac{1}{2}+it\right).$$

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The function $\Xi_{\mathbb{F}}(t)$ is even and real for all real values of t.

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Then we define a function $f(t) := |\phi(it)|^2 = \phi(it)\phi(-it),\ t \in \mathbb{R}$, where $\phi(s)$ is an analytic function with $\overline{\phi(s)} = \phi(\bar{s})$.

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Then we define a function $f(t):=|\phi(it)|^2=\phi(it)\phi(-it),\ t\in\mathbb{R}$, where $\phi(s)$ is an analytic function with $\overline{\phi(s)}=\phi(\bar{s})$. Now, we examine the following integral, for $z\in\mathbb{C}$,

$$\Phi(z)=\int_0^\infty f(t)\Xi_{\mathbb{F}}(t)\cos(zt)\mathrm{d}t.$$

Proof continued...

We write $\cos(zt) = \frac{e^{izt}+e^{-izt}}{2}$ and substitute $e^z = y$ to see

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$$\begin{aligned}\Phi(z) &= \frac{1}{2}\int_0^\infty f(t)\Xi_{\mathbb{F}}(t)y^{it}\mathrm{d}t + \frac{1}{2}\int_{-\infty}^0 f(t)\Xi_{\mathbb{F}}(t)y^{it}\mathrm{d}t \\ &= \frac{1}{2}\int_{-\infty}^\infty f(t)\Xi_{\mathbb{F}}(t)y^{it}\mathrm{d}t \\ &= \frac{1}{2}\int_{-\infty}^\infty \phi(it)\phi(-it)\xi_{\mathbb{F}}\left(\frac{1}{2} + it\right) y^{it}\mathrm{d}t.\end{aligned}$$

Proof continued ...

Letting $\frac{1}{2} + it = s$ and $\phi(s) = \frac{1}{s+\frac{1}{2}}$ in the above integral and using (2), one can see that

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Since $y=e^z$ and $f(t)=\phi(it)\phi(-it)=\frac{1}{t^2+\frac{1}{4}}$, so we have

$$\begin{aligned}\Phi(z) &= \int_0^\infty\frac{\Xi_{\mathbb{F}}(t)}{t^2+\frac{1}{4}}\cos(zt)\mathrm{d}t \qquad (4)\\ &= -\frac{\pi}{2}\left[e^{-\frac{z}{2}}W_{\mathbb{F},1}(e^{-2z})+C_{\mathbb{F}}2^{r_1}\left(e^{-\frac{z}{2}}+e^{\frac{z}{2}}\right)\right]. \qquad (5)\end{aligned}$$

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Now we mention the following exact evaluation of an integral, where the integrand is related to the functional equation of the Riemann zeta function, namely,

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)y^{s}\mathrm{d}s = W_1\left(\frac{1}{y^2}\right)-(1+y),$$

where $W_1(x)$ is the Jacobi theta function.

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Lemma (Bansal, M.)

Let $\Omega_{\mathbb{F}}(s)$ be one side of the functional equation of $\zeta_{\mathbb{F}}(s)$ and $W_{\mathbb{F},1}(x)$ be the number field analogue of Jacobi theta function.

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$$\int_0^\infty \frac{\Xi_{\mathbb{F}}(t)}{t^2 + \frac{1}{4}} \cosh(\alpha t) \mathrm{d}t = -\frac{\pi}{2}\left[C_{\mathbb{F}} 2^{r_1}\left(e^{\frac{-i\alpha}{2}} + e^{\frac{i\alpha}{2}}\right) + e^{\frac{i\alpha}{2}} W_{\mathbb{F},1}\left(e^{2i\alpha}\right)\right]$$

$$= -\frac{\pi}{2}\left[2C_{\mathbb{F}} 2^{r_1}\left(\frac{e^{\frac{i\alpha}{2}} + e^{-\frac{i\alpha}{2}}}{2}\right) + e^{\frac{i\alpha}{2}} W_{\mathbb{F},1}(e^{2i\alpha})\right]$$

$$= -\pi C_{\mathbb{F}} 2^{r_1} \cos\left(\frac{\alpha}{2}\right) - \frac{\pi e^{\frac{i\alpha}{2}}}{2} W_{\mathbb{F},1}(e^{2i\alpha}).$$

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$$\begin{aligned}\int_0^\infty \frac{\Xi_{\mathbb{F}}(t)}{(t^2+\frac{1}{4})}t^{2n}\cosh(\alpha t)\mathrm{d}t &= (-1)^{n+1}\frac{\pi C_{\mathbb{F}}2^{r_1}\cos\left(\frac{\alpha}{2}\right)}{2^{2n}}\\ &\quad -\frac{\pi}{2}\frac{\mathrm{d}^{2n}}{\mathrm{d}\alpha^{2n}}\left(e^{\frac{i\alpha}{2}}W_{\mathbb{F},1}(e^{2i\alpha})\right).\end{aligned}$$

Bound for $\Xi_{\mathbb{F}}(t)$

Note that the above identity is valid for $|\alpha| < \frac{\pi d}{4}$ since the factor $W_{\mathbb{F},1}(x)$ converges for $|\operatorname{Arg}(x)| < \frac{\pi d}{2}$. One can easily prove that

$$\Xi_{\mathbb{F}}(t) = O\left(|t|^A D^{\frac{1}{4}+\epsilon} e^{-\frac{\pi d|t|}{4}}\right)$$

for some positive A, by using the bounds $\zeta_{\mathbb{F}}\left(\frac{1}{2}+it\right) = O(|t|^{\frac{d}{4}+\epsilon} D^{\frac{1}{4}+\epsilon})$ and $|\Gamma(\frac{1}{2}+it)| = O(e^{-\frac{\pi}{2}|t|})$ as $|t| \to \infty$.

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Proof continued...

From Corollary (1), we have

$$W_{\mathbb{F},1}\left(e^{-2i\alpha}\right) = e^{i\alpha} W_{\mathbb{F},1}(e^{2i\alpha}), \quad \text{for } |\alpha| < \frac{\pi d}{4}.$$

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$$\int_0^\infty \frac{\Xi_{\mathbb{F}}(t)}{\left(t^2 + \frac{1}{4}\right)} t^{2n} \cosh\left(\frac{\pi t}{2}\right) \mathrm{d}t = (-1)^{n+1} \frac{C_{\mathbb{F}} 2^{r_1} \pi \cos\left(\frac{\pi}{4}\right)}{2^{2n}}, \quad \forall n.$$

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$$\begin{aligned}\int_T^\infty \frac{\Xi_{\mathbb{F}}(t)}{\left(t^2 + \frac{1}{4}\right)} t^{2n} \cosh\left(\frac{\pi t}{2}\right) \mathrm{d}t &< -\int_0^T \frac{\Xi_{\mathbb{F}}(t)}{\left(t^2 + \frac{1}{4}\right)} t^{2n} \cosh\left(\frac{\pi t}{2}\right) \mathrm{d}t \\ &< \ell \int_0^T t^{2n} \mathrm{d}t < \ell\, T^{2n+1}.\end{aligned}$$

Proof Continued...

We also have

$$\begin{aligned}\ell\, T^{2n+1} &> \int_T^\infty \frac{\Xi_{\mathbb{F}}(t)}{\left(t^2+\frac{1}{4}\right)} t^{2n} \cosh\left(\frac{\pi t}{2}\right) \mathrm{d}t \\ &> \int_{\delta T}^{T(\delta+1)} \frac{\Xi_{\mathbb{F}}(t)}{\left(t^2+\frac{1}{4}\right)} t^{2n} \cosh\left(\frac{\pi t}{2}\right) \mathrm{d}t \\ &> c\int_{\delta T}^{T(\delta+1)} t^{2n} \mathrm{d}t \\ &> c\delta^{2n} T^{2n+1}.\end{aligned}$$

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Thus, $\ell > \delta^{2n} c$, which fails to hold for sufficiently large n, as $\delta > 1$. Hence our assumption is wrong. Therefore, $\Xi_{\mathbb{F}}(t)$ must change sign infinitely often, which implies that it has infinitely many real zeros and so has $\zeta_{\mathbb{F}}(s)$ on the critical line.

Future directions

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He also obtained a zero-density estimate, for $\frac{1}{2} < \sigma < 1$,

$$N(\sigma, T) := \#\{\rho = \beta + i\gamma : \beta > \sigma,\ 0 < \gamma \le T\} \ \ll\ T^{1-\frac{1}{4}\left(\sigma-\frac{1}{2}\right)}\log T.$$

- D. R. Bansal and B. Maji, *Number field analogue of Jacobi theta relation and zeros of Dedekind zeta function on Re(s)=1/2*, submitted, arXiv:2507.18121, 2025.
- B. C. Berndt, *On the zeros of a class of Dirichlet series II*, Ill. J. Math., **14** (1970), 678–691.
- A. Dixit, S. Gupta, and A. Vatwani, *A Modular Relation Involving Non-trivial Zeros of the Dedekind Zeta Function, and the Generalized Riemann Hypothesis*, J. Math. Anal. Appl., **515**(2), p.126435, 2022.
- W. L. Ferrar, *Some Solutions of The Equation* $F(t) = F(t^{-1})$, J. London Math. Soc., 1**(2)**, 99–103, 1936.
- G. H. Hardy, *Sur les zéros de la fonction* $\zeta(s)$ *de Riemann*, C. R. Acad. Sci. Paris, **158** (1914), 1012–1014.

Thank you for your attention!