HARMONY IN ALGEBRAIC GEOMETRY

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April 17, 2024

The goal of this talk is to explain the terms and the motivation for the following theorem.

Theorem (Can-Diaz 2023)

Let Fl_n denote the variety of full flags of subspaces of the complex vector space $\mathbb{C}^n$. Then, for $n \geq 5$, we have

1. *The number of smooth nearly toric Schubert varieties in Fl_n is*

$$r_n := (n-2)F_{2(n-2)}.$$

2. *The number of singular nearly toric Schubert varieties in Fl_n is*

$$b_n := \frac{2(2n-7)F_{2n-8} + (7n-23)F_{2n-7}}{5}.$$

Here, F_m denotes the m-th Fibonacci number.

The sequence of Fibonacci numbers $(F_m)_{m\geq 0}$ is defined by the initial conditions

$$F_0 = 0 \quad \text{and} \quad F_1 = 1$$

and the recurrence

$$F_m = F_{m-1} + F_{m-2} \quad \text{for } m \geq 2.$$

Remark

The book Liber Abaci (1202) of Leonardo Pisano appears to be the first published account of Fibonacci numbers in Europe. Nevertheless, these numbers were appreciated in India more than a thousand years earlier than the publication of Liber Abaci. In fact, it is widely accepted that the Indian poet and mathematician Acharya Pingala (450 BC-200 BC) was already working with the Fibonacci numbers to study the patterns in Sanskrit poetry.

It is not surprising that these days we know a lot about the Fibonacci numbers. Let me write here for completeness some useful facts about them.

Let $F(x)$ denote the generating series $\sum_{m\geq 0} F_m x^m$. It follows from the basic recurrence of the Fibonacci numbers that

$$F(x) = \frac{x}{1-x-x^2}.$$

Let $\varphi := \frac{1+\sqrt{5}}{2}$ and $\psi := \frac{1-\sqrt{5}}{2}$. Then we see that

$$\frac{x}{1-x-x^2} = \frac{x}{(1-\varphi x)(1-\psi x)} = \frac{1}{1-\varphi x} - \frac{1}{1-\psi x}.$$

It follows from the power series expansion that

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}}.$$

Note that the number φ is widely accepted to be the most eye-pleasing ratio

$$\frac{\text{the length of the long-side}}{\text{the length of the short side}}$$

of an A4 size paper. Indeed, if we have the square

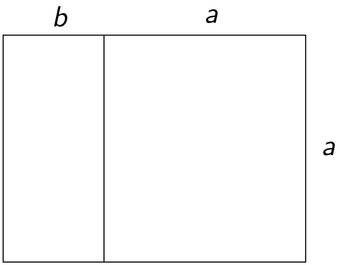

Figure: Rectangle with dimensions a and $a + b$.

Then the ratio $\frac{a+b}{a} = \frac{a}{b}$ yields $\frac{a}{b} = \varphi = \frac{1+\sqrt{5}}{2}$.

Next, I would like to discuss patterns in permutations.

A permutation of the set $\{1, 2, \ldots, n\}$ is a self-bijection of the set $\{1, 2, \ldots, n\}$. The set of all permutations of $\{1, \ldots, n\}$ is denoted by S_n.

Let us call the elements of the set $\{1, 2, \ldots, n\}$ the letters so that when we list the values of the permutation we get a word:

$$\sigma_1 \sigma_2 \ldots \sigma_n,$$

where $\sigma_j = \sigma(j)$ for $j = 1, \ldots, n$.

A useful "statistic" that is attached to a permutation σ is the total number of out-of-order values that it takes:

$$\mathrm{inv}(\sigma) = |\{(i, j) \mid 1 \leq i < j \leq n \text{ and } \sigma_j < \sigma_i\}.$$

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Let n and k be two positive integers with $k \leq n$. Let τ be a permutation of $\{1, \ldots, k\}$. We call τ the *pattern* for the moment.

This pattern τ is said to occur in a permutation σ of $\{1, \ldots, n\}$ if there are integers $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that for all $1 \leq r < s \leq k$ we have

$$\tau(r) < \tau(s) \iff \sigma(i_r) < \sigma(i_s).$$

Example

In S_4, there are exactly 10 permutations that contain the pattern $\tau = 312$:

$$\begin{array}{ccccc} 1\underline{423} & 2\underline{413} & \underline{312}4 & \underline{31}4\underline{2} & 3\underline{412} \\ \underline{412}3 & \underline{413}2 & \underline{4}2\underline{13} & \underline{423}1 & 4\underline{312} \end{array}$$

Note that the entries of the pattern τ need not be adjacent in σ. Also, the pattern τ may appear multiple times in the same permutation.

Our previous example shows that there are 24-10= 14 permutations in S_4 without the pattern 312 in them. This phenomenon is called the *pattern avoidence*. If a pattern τ does not appear in a permutation σ, then we will say that σ is a *τ-avoiding permutation*.

Theorem (Knuth 1962)

The number of 312-avoiding permutations of $\{1, \dots, n\}$ is given by the n-th Catalan number, that is,

$$c_n := \frac{1}{n+1}\binom{2n}{n}.$$

Note: c_n gives the number of Dyck paths of size n. Here, a *Dyck path of size n* is a lattice path in $\mathbb{Z}^2$ that starts at $(0, 0)$ and ends at (n, n) moving with $(1, 0)$ or $(0, 1)$ steps, while staying weakly above the main diagonal $x = y$.

Next, we will discuss flag varieties. We need the basic group theoretic notation. This will be useful when we introduce "spherical varieties."

$$\begin{aligned} \mathbf{GL}(n,\mathbb{C}) &:= \{A \in \mathbf{Mat}(n,\mathbb{C}) \mid \det A \neq 0\} \\ \mathbf{B}(n,\mathbb{C}) &:= \{A \in \mathbf{Mat}(n,\mathbb{C}) \mid A \text{ is upper triangular}\}. \end{aligned}$$

Definition

Let Fl_n denote the set of sequences $(V_i)_{i=0}^n$ where V_j (for $j \in \{0,\ldots,n-1\}$) is a vector subspace of V_{j+1}, and $\dim V_j = j$. Such a sequence is called a *full flag*.

Since $\mathbf{GL}(n,\mathbb{C})$ acts transitively on Fl_n, and since $\mathbf{B}(n,\mathbb{C})$ stabilizes the standard full flag

$$0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_n \rangle,$$

we identify Fl_n with the coset space $\mathbf{GL}(n,\mathbb{C})/\mathbf{B}(n,\mathbb{C})$.

We can represent each permutation as a 0/1 matrix. For example,

$$51324 \rightsquigarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

With this matrix representation, we can state the remarkable relationship between the structure of $\mathbf{GL}(n,\mathbb{C})$ and that of Fl_n via $\mathbf{B}(n,\mathbb{C})$.

Proposition (A special case of the Bruhat-Chevalley decomposition)

$$\mathbf{GL}(n,\mathbb{C}) = \bigsqcup_{\sigma \in S_n} \mathbf{B}(n,\mathbb{C})\ \sigma\ \mathbf{B}(n,\mathbb{C}).$$

This theorem is a version of the $LDPU$-decomposition for matrices, where L (resp. U) stands for a lower triangular (resp. upper triangular) matrix with 1's along its main diagonal, D stands for a diagonal matrix, and P stands for a permutation matrix.

Each double coset $\mathbf{B}(n,\mathbb{C})\ \sigma\ \mathbf{B}(n,\mathbb{C})$, where $\sigma \in S_n$, is a subset of $\mathbf{GL}(n,\mathbb{C})$, which is a subset of $\mathbf{Mat}(n,\mathbb{C})$. Since $\mathbf{Mat}(n,\mathbb{C})$ is essentially the same space as $\mathbb{C}^{n^2}$, we may use restrict the topology from $\mathbb{C}^{n^2}$ to our double cosets. In particular, we can take the closure of $\mathbf{B}(n,\mathbb{C})\ \sigma\ \mathbf{B}(n,\mathbb{C})$ in $\mathbb{C}^{n^2}$, and then send this closed set to Fl_n.

Definition

The *Schubert variety* associated with the permutation $\sigma \in S_n$ is the image of the closure $\overline{\mathbf{B}(n,\mathbb{C})\ \sigma\ \mathbf{B}(n,\mathbb{C})}$ in $\mathbf{GL}(n,\mathbb{C})/\mathbf{B}(n,\mathbb{C})$ under the canonical map

$$\pi : \mathbf{GL}(n,\mathbb{C}) \to \mathbf{GL}(n,\mathbb{C})/\mathbf{B}(n,\mathbb{C}).$$

We will denote it by X_σ.

Remarkably, the dimension of the Schubert variety X_σ is given by the combinatorial inversion number function:

$$\dim X_\sigma = \mathrm{inv}(\sigma).$$

The next concept that we want to discuss can be motivated by some fundamental concepts of harmonic analysis, hence the title of our talk.

Let us consider the two dimensional sphere

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

The special orthogonal group $\mathbf{SO}(3,\mathbb{R})$ acts on S^2 by rotations. The stabilizer of the south pole is isomorphic to the subgroup $\mathbf{SO}(2,\mathbb{R})$. In other words, we have the following identification:

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A *harmonic function* $f : \mathbb{R}^3 \to \mathbb{C}$ is a solution of the Laplace equation

$$\frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} + \frac{\partial^2 f}{\partial^2 z} = 0.$$

We denote by $\mathcal{H}(S^2)$ the space of all harmonic functions on S^2. Then we have,

$$\mathcal{H}(S^2) = \bigoplus_{j \geq 0} \mathcal{H}_j(S^2),$$

where $\mathcal{H}_j(S^2)$ denotes the vector space spanned by all harmonic homogeneous polynomial functions of degree j defined on S^2.

The summands of this decomposition give all inequivalent, irreducible representations of $\mathbf{SO}(n, 3)$. This is a typical example of a "multiplicity-free" phenomenon.

One lesson we learn from the spherical harmonics is that certain homogeneous spaces (i.e., coset spaces) might carry exceptional representation theoretic information.

Building upon this observation, we will now consider a broader concept, but within the context of general linear groups.

Definition

Let G be a product of finitely many general linear groups of various sizes. Let X be a (normal) algebraic variety on which G acts algebraically. If the subgroup of L consisting of all upper triangular matrices has an open orbit in X, then X is called a *spherical G-variety*.

Proposition (Kimelfeld-Vinberg)

Assume that X is affine. Then X is a spherical G-variety if and only if the ring of regular functions on X has a decomposition into inequivalent irreducible G-representations, $\mathbb{C}[X] = \bigoplus_{\nu} V(\nu)$.

An important result, obtained separately by Brion and Vinberg is the following.

Theorem

Let X be a (normal) normal variety on which G acts algebraically. Let B_G denote the subgroup of all upper triangular matrices in G. Then B_G-has an open orbit in X if and only if B_G has only finitely many orbits in X.

Example

Let $G := \prod_{i=1}^{n} \mathbf{GL}(1, \mathbb{C}) \cong (\mathbb{C}^*)^n$. Then, naturally, $B_G = G$. In this case, X is a spherical G-variety if and only if $(\mathbb{C}^*)^n$ has an open orbit in X. These varieties are precisely the *toric varieties*.

Let us reformulate the definition of the spherical varieties.

Definition

Let X be a (normal) variety on which G acts algebraically. The *G-complexity* of the action of G is defined by

$$c_G(X) := \min\{\mathrm{codim}(B_G \cdot x) \mid x \in X\}.$$

The novel definition of our work is defined as follows.

Definition

A spherical G-variety X is called *nearly toric* if the T-complexity of X is 1. Here, T stands for the subgroup of G consisting of all diagonal matrices.

In other words, X is a nearly toric variety iff $c_G(X) = 0$ and $c_T(X) = 1$.

Example

Let X denote the space of degenerate 4×4 skew-symmetric matrices. Then

$$X \cong \bigwedge^2 \mathbb{C}^4 \setminus \mathbf{GL}(4, \mathbb{C}) \cdot v,$$

where $v \in \bigwedge^2 \mathbb{C}^4$ is a 2-form in general position. It is well-known that the following action is a spherical action:

$$\begin{aligned} \mathbf{GL}(4, \mathbb{C}) \times X &\longrightarrow X \\ (A, B) &\mapsto ABA^\top. \end{aligned}$$

It is also easy to see that the restriction of the action of $\mathbf{GL}(4, \mathbb{C})$ to its maximal torus T has (maximal) 4 dimensional orbits. Since $\dim X = 5$, we see that

$$c_T(X) = 5 - 4 = 1.$$

Therefore, X is a nearly toric $\mathbf{GL}(4, \mathbb{C})$-variety.

We now sketch the proof of our theorem.

There is a pattern avoidance characterization of the permutations $w \in S_n$ such that X_w is a spherical L-variety, where L is the Levi subgroup of the stabilizer,

$$L \subset \mathrm{Stab}(X_w) := \{g \in \mathbf{GL}(n, \mathbb{C}) \mid gX_w = X_w\}.$$

The Levi subgroup is isomorphic to a product of the form $\prod_{i=1}^{r} \mathbf{GL}(n_i, \mathbb{C})$, where $\sum_{i=0}^{r} n_i = n$.

Lemma (Gao-Hodges-Yong Conjecture, proven by Gaetz, 2022)

The Schubert variety X_w is a spherical L-variety iff w avoids the following 21 patterns:

$$\mathscr{P} := \begin{Bmatrix} 24531 & 25314 & 25341 & 34512 & 34521 & 35412 & 35421 \\ 42531 & 45123 & 45213 & 45231 & 45312 & 52314 & 52341 \\ 53124 & 53142 & 53412 & 53421 & 54123 & 54213 & 54231 \end{Bmatrix}.$$

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The T-complexity 1 Schubert varieties have pattern avoidance characterizations as well.

Lemma (Lee-Masuda-Park 2021)

Let $w \in S_n$. Then

1. *X_w is smooth and $c_T(X_w) = 1$ iff w contains the pattern 3412 exactly once and avoids 321*
2. *X_w is singular and $c_T(X_w) = 1$ iff w contains 321 exactly once and avoids 3412.*

Instrumental in the this result of Lee-Park-Masuda is a work of Daly (around 2010).

Lemma (Daly 2010)

Let $\mathcal{A}_n$ denote the set of permutations $w \in S_n$ such that w avoids 3412 contains 321 only once. Let $\mathcal{M}_n$ denote the set of permutations $w \in S_n$ such that w avoids 3412 contains 321 only once and w contains 25314. Then $|\mathcal{A}_n| = |\mathcal{M}_{n+2}|$. Furthermore, the generating series of $a_n := |\mathcal{A}_n|$ is given by

$$\sum_{n\geq 3} a_n x^n = \frac{x^3}{(1-3x+x^2)^2}.$$

Another result that we used is the following important theorem of Lakshmibai and Sandhya.

Lemma (Lakshmibai-Sandhya 1990)

Let $w \in S_n$. The Schubert variety X_w is smooth if and only if w avoids 3412 and 4231.

All of these facts allowed us to prove the following theorem that led us to the enumeration of nearly toric Schubert varieties.

Theorem (Can-Diaz)

Let $w \in S_n$. Then X_w is a nearly toric Schubert variety iff one of the following holds:

1. *X_w is singular; w contains the pattern 3412 exactly once and avoids the pattern 321.*
2. *X_w is smooth; w contains the pattern 321 exactly once and avoids the following patterns:*

$$\mathscr{P}' := \begin{Bmatrix} 24531 & 25314 & 25341 & 34521 & 35421 \\ 42531 & 52314 & 52341 & 54213 & 54231 \\ 53124 & 53142 & 53421 & 54123 & 3412 \end{Bmatrix}.$$

There are many calculations and intermediate steps to discuss, but we've omitted them here.

I want to mention one more combinatorially interesting result that goes well with the combinatorial aspects of our talk.

Theorem (Can-Diaz 2023)

Let $A \subset S_n$ denote the set 312-avoding permutations. For $w \in A$, let π denote the corresponding Dyck path of size n. Then X_w is a spherical Schubert variety iff π is a spherical Dyck path.

Here, we call a Dyck path π a spherical Dyck path if

- every connected component of π on the first diagonal is either an elbow or a ledge, or
- every connected component of π on the second diagonal is an elbow, or a ledge whose E extension is the initial step of a connected component of π on the first diagonal.

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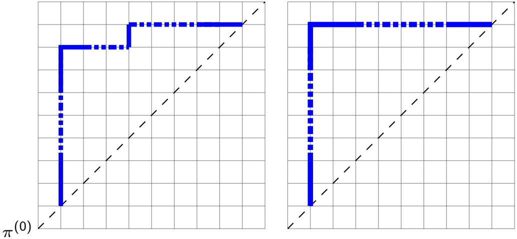
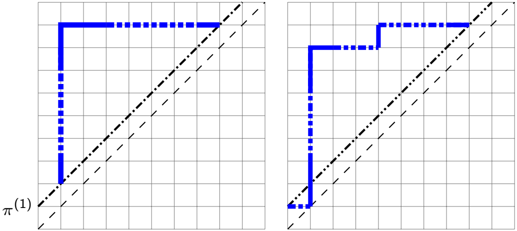


(a) A ledge or an elbow of $\pi^{(0)}$.



(b) An elbow or a ledge of $\pi^{(1)}$.

Figure: Spherical Dyck paths

THIS IS THE END, UNTIL NEXT TIME.

THANK YOU!