Metric entropy for bounded variation functions

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Overview

- **Introduction**
- 1-D BV functions
- Metric entropy for 1-D BV functions
- An application

Motivation

Consider the first order PDE

$$F(t, x, u, Du) = 0, \;\; u(0, \cdot) = u_0 \; . \tag{1}$$

Given any bounded set of initial data C and $T > 0$, denote by

$$S_T(C) := \{u(T, \cdot) \mid u \text{ solves (1) and } u(0, \cdot) \in C\}$$

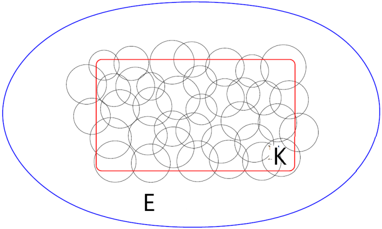
the set of all solutions of (1) with initial data $u_0 \in C$ at time T .

Main Question: Is it possible to measure $S_T(C)$?

Lax's suggestion: Use Kolmogorov ε-entropy

Metric entropy or $\varepsilon - entropy$

Let (E, ρ) be a metric space and K be a totally bounded subset of E. For $\varepsilon > 0$, let $\mathcal{N}_\varepsilon(K\big|E)$ be the minimal number of sets in an ε-covering of K.



The ε-entropy of K is defined as

$$\mathcal{H}_\varepsilon(K|E) \ = \ \log_2 \mathcal{N}_\varepsilon(K|E) \ .$$

In other words, it is the minimal number of binary digits (bits) needed to represent a point with accuracy ε.

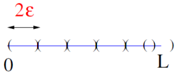
History and Motivation

Introduced by Kolmogorov and Tikhomirov in 1959.

- Application to Probability
- Application to Statistics
- Application to Information Theory
- Application to Dynamical Systems and Biological Models
- Application to PDEs
- Application to Numerical Analysis

Examples of ε-entropy estimates

- $E = \mathbb{R},\ K = [0, L]$



$$\mathcal{N}_\varepsilon([0, L]|\mathbb{R}) \approx \frac{L}{2\varepsilon} \text{ and } \mathcal{H}_\varepsilon([0, L]|\mathbb{R}) \approx -log_2(\varepsilon)$$

- $E = \mathbb{R}^d$, $\rho(x, y) = \|x - y\|$ and $K = B(0, r)$ for some $r > 0$

$$d \cdot \log_2\left(\frac{r}{\varepsilon}\right) \ \leq\ \mathcal{H}_\varepsilon\left(B(0, r)\middle|\mathbb{R}^d\right) \ \leq\ d \cdot \log_2\left(\frac{2r}{\varepsilon} + 1\right)$$

- **d-dimensional Lipschitz functions**: Let F_d be the set of L-Lipschitz functions (w.r.t $\|\cdot\|_\infty$) from $[0, 1]^d$ to $[0, 1]$. Then

$$\mathcal{H}_\varepsilon(F_d|\mathbf{L}^1([0, 1]^d, [0, 1])) \ \approx\ \left(\frac{L}{\varepsilon}\right)^d.$$

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Bounded variation functions in 1-D

Definition

Given $f : [a, b] \to \mathbb{R}$, let $P = \{a = x_0, x_1, \cdots, x_n = b\}$ be a partition of $[a, b]$. We say that f **has bounded total variation** or, $f \in BV([a, b], \mathbb{R})$ if it holds that

$$\sup_{P \in \mathcal{P}[a,b]} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \; < \; \infty \; .$$

In that case, the **total variation of f on $[a, b]$** is defined as

$$TV(f, [a, b]) := \sup_{P \in \mathcal{P}[a,b]} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \; .$$

Important results on BV functions in 1-D

- If $f : [a, b] \to \mathbb{R}$ is **monotone** then f is a BV function and

$$TV(f, [a, b]) \;=\; |f(b) - f(a)| \;.$$

- If $f : [a, b] \to \mathbb{R}$ is differentiable having a bounded derivative, then it is a BV function.

- If $f : [a, b] \to \mathbb{R}$ is a BV function, it can be expressed as a **difference of two nondecreasing functions**.

Given $L, M, V > 0$, we define

$$\mathcal{B}_{[L,M,V]} \;=\; \left\{ f : [0, L] \to [0, M] \;\middle|\; TV(f, [0, L]) \;\leq\; V \right\} .$$

Helly's Theorem

$\mathcal{B}_{[L,M,V]}$ is compact in $\mathbf{L}^1([0, L], [0, M])$.

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Entropy in $\mathbf{L}^1$ for a class of nondecreasing functions

For $L, M > 0$, define

$$\mathcal{I}_{L,M} := \{w : [0, L] \to [0, M] \mid w \text{ is nondecreasing}\} .$$

Lemma (De Lellis and Golse, *Comm. Pure Appl. Math.*, 2005)

For $0 < \varepsilon \leq \frac{LM}{6}$, *it holds that*

$$\mathcal{H}_\varepsilon \left(\mathcal{I}_{L,M} \mid \mathbf{L}^1([0, L])\right) \ \leq \ 4 \cdot \left[\frac{LM}{\varepsilon}\right] .$$

Main Result

Given $L, M, V > 0$, we define

$$\mathcal{B}_{[L,M,V]} \;=\; \left\{ f : [0,L] \to [0,M] \;\middle|\; TV(f,[0,L]) \;\le\; V \right\} .$$

Theorem (D- and Nguyen, *J. Math. Anal. Appl.*, 2018)

For all $0 < \varepsilon < \frac{L(M+V)}{6}$, *it holds that*

$$\mathcal{H}_{\varepsilon}\left(\mathcal{B}_{[L,M,V]} | \mathbf{L}^1([0,L]) \right) \;\le\; 8 \cdot \left[\frac{L(M+V)}{\varepsilon} \right] .$$

Proof of the theorem (I)

Step 1: For any $f \in \mathcal{B}_{[L,M,V]}$, define the function $V_f(x) = TV(f, [0, x])$.

Step 2: Decompose f as $f(x) = f^+(x) - f^-(x)$ for all $x \in [0, L]$ where

$$f^- = \frac{V_f - f}{2} + \frac{M}{2}$$

is a nondecreasing function from $[0, L]$ to $\left[0, \frac{V+M}{2}\right]$ and

$$f^+ = \frac{V_f + f}{2} + \frac{M}{2}$$

is a nondecreasing function from $[0, L]$ to $\left[\frac{M}{2}, \frac{V+2M}{2}\right]$. Define

$$\mathcal{I} := \left\{ g : [0, L] \to \left[0, \frac{V+M}{2}\right] \;\middle|\; g \text{ is nondecreasing} \right\}.$$

Proof of the theorem (II)

Step 3: Observe that

$$\mathcal{B}_{[L,M,V]} \subseteq \left(\mathcal{I} + \frac{M}{2}\right) - \mathcal{I} := \left\{ g - h \,\middle|\, g \in \mathcal{I} + \frac{M}{2} \quad \text{and} \quad h \in \mathcal{I} \right\}. \tag{2}$$

Then for any $\varepsilon > 0$, we claim that

$$\mathcal{N}_{\varepsilon}\left(\mathcal{B}_{[L,M,V]} \mid \mathbf{L}^1([0,L])\right) \leq \left[\mathcal{N}_{\frac{\varepsilon}{2}}(\mathcal{I} \mid \mathbf{L}^1([0,L]))\right]^2 .$$

Proof of the claim: Let $\mathcal{G}_{\frac{\varepsilon}{2}}$ be an $\frac{\varepsilon}{2}$-covering of $\mathcal{I}$ in $\mathbf{L}^1([0,L])$. By definition, this means

$$\mathcal{I} \subseteq \bigcup_{\mathcal{E} \in \mathcal{G}_{\frac{\varepsilon}{2}}} \mathcal{E} \quad \text{and} \quad \text{diam}(\mathcal{E}) = \sup_{h_1, h_2 \in \mathcal{E}} \|h_1 - h_2\|_{\mathbf{L}^1([0,L])} \leq \varepsilon.$$

Thus, (2) implies $\mathrm{B}_{[L,M,V]} \subseteq \bigcup_{(\mathcal{E}_1, \mathcal{E}_2) \in \mathcal{G}_{\frac{\varepsilon}{2}} \times \mathcal{G}_{\frac{\varepsilon}{2}}} \left[\left(\mathcal{E}_1 + \frac{M}{2}\right) - \mathcal{E}_2\right]$.

Proof of the theorem (III)

For any two functions

$$f_i \;=\; g_i - h_i \;\in \left(\mathcal{E}_1 + \frac{M}{2}\right) - \mathcal{E}_2 \qquad \text{for } i = 1, 2\,,$$

we have

$$\begin{aligned}\|f_1 - f_2\|_{\mathbf{L}^1([0,L])} \;&\leq\; \|g_1 - g_2\|_{\mathbf{L}^1([0,L])} + \|h_1 - h_2\|_{\mathbf{L}^1([0,L])} \\ &\leq\; \mathrm{diam}\left(\mathcal{E}_1 + \frac{M}{2}\right) + \mathrm{diam}(\mathcal{E}_2) \;\leq\; \varepsilon + \varepsilon \;=\; 2\varepsilon\end{aligned}$$

and this implies that

$$\mathrm{diam}\left[\left(\mathcal{E}_1 + \frac{M}{2}\right) - \mathcal{E}_2\right] \;\leq\; 2\varepsilon\,.$$

Therefore,

$$\mathcal{N}_\varepsilon\left(\mathcal{B}_{[L,M,V]} \;\middle|\; \mathbf{L}^1([0.L])\right) \;\leq\; \left[\mathcal{N}_{\frac{\varepsilon}{2}}(\mathcal{I} \mid \mathbf{L}^1([0, L]))\right]^2 .$$

Proof of the theorem (IV)

Step 4: From Step 3, by the definition of metric entropy it follows that

$$\mathcal{H}_{\varepsilon}\left(\mathcal{B}_{[L,M,V]}\,\Big|\,\mathbf{L}^1([0,L])\right) \;\leq\; 2\cdot\mathcal{H}_{\frac{\varepsilon}{2}}\left(\mathcal{I}\,\Big|\,\mathbf{L}^1([0,L])\right)\,. \tag{3}$$

Step 5: Applying De Lellis and Golse's entropy estimate for $\mathcal{I}$, , we get

$$\mathcal{H}_{\frac{\varepsilon}{2}}\left(\mathcal{I}\mid\mathbf{L}^1([0,L])\right) \;\leq\; 4\cdot\left\lfloor\frac{L(M+V)}{\varepsilon}\right\rfloor\,,$$

for any $0<\varepsilon<\frac{L(M+V)}{6}$ and then (3) yields

$$\mathcal{H}_{\varepsilon}\left(\mathcal{B}_{[L,M,V]}|\mathbf{L}^1([0,L])\right) \;\leq\; 8\cdot\left[\frac{L(M+V)}{\varepsilon}\right]\,.$$

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Metric entropy for multi-dimensional BV functions

Given $u \in \mathbf{L}^1(\Omega, \mathbb{R})$ where $\Omega \subset \mathbb{R}^d$ is open, we say that u is of bounded variation on Ω, i.e., $u \in BV(\Omega, \mathbb{R})$ if

$$\int_\Omega u \cdot \frac{\partial \varphi}{\partial x_i}\, dx \;=\; -\int_\Omega \varphi dD_i u \qquad \text{for all } \varphi \in \mathcal{C}_c^1(\Omega, \mathbb{R}),\; i \in \{1, 2, ..., d\}$$

for some finite Radon measure $Du = (D_1 u, D_2 u, ..., D_d u)$.

Let the set of BV functions be denoted by

$$\mathcal{F}_{[L,M,V]} \;=\; \left\{ u \in \mathbf{L}^1([0,L]^d, \mathbb{R}) \;\middle|\; \|u\|_{\mathbf{L}^\infty([0,L]^d)} \leq M,\; TV(u, [0,L]^d) \leq V \right\}.$$

Theorem (D - and Nguyen, *J. Math. Anal. Appl.*, 2018)

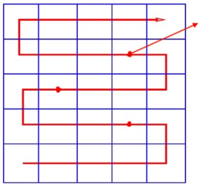
Given $L, M, V > 0$, for every $0 < \varepsilon < \frac{ML^d}{8}$, it holds that

$$\frac{\log_2(e)}{8} \cdot \left\lfloor \frac{LV}{2^{d+2}\varepsilon} \right\rfloor^d \;\leq\; \mathcal{H}_\varepsilon\left(\mathcal{F}_{[L,M,V]} \mid \mathbf{L}^1([0,L]^d, \mathbb{R})\right)$$

$$\leq\; \frac{1}{\varepsilon^d}\left[\frac{8}{\sqrt{d}}\left(4\sqrt{d}LV\right)^d + \left(\frac{2^{d+7}V}{M} + 8\right) \cdot \left(\frac{ML^d}{8}\right)^d\right].$$

Sketch of the Proof

- Upper estimate:

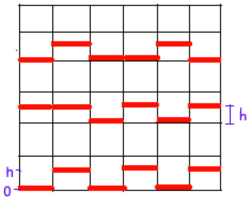


$$u_\iota = \frac{1}{\mathrm{Vol}(\square_\iota)} \int_{\square_\iota} u(x)\, dx$$

$$\tilde{u}(x) = \begin{cases} u_\iota & \text{for all } x \in \mathrm{int}(\square_\iota), \\ 0 & \text{for all } x \in \bigcup_{\iota \in \{0,1,\ldots,N-1\}^n} \partial\square_\iota . \end{cases}$$

- Lower estimate:

Step function :



Thank you
for your attention!