Borwein integrals and Bernoulli Sophomore's Dream identity, from continuous to discrete

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Introduction

Boas and Pollard [1]

$$\sum_{n\in\mathbb{Z}}\left(\frac{\sin\left[\alpha\left(n+z\right)\right]}{n+z}\right)^2=\int_{-\infty}^{\infty}\left(\frac{\sin\left[\alpha\left(x+z\right)\right]}{x+z}\right)^2 dx=\frac{\pi}{\alpha},\quad 0<\alpha<\pi$$

[1] R.P. Boas Jr and H. Pollard, Continuous analogs of series, The American Mathematical Monthly, 80, 1, 18-25, 1973

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Ramanujan

$$\sum_{k\geq 1}k^{-k}=\int_0^1 x^{-x}dx$$

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Introduction

A guess by Ramanujan (Vol IV, p.310, Entry 22)

Entry 22 (p. 318). *Formally,*

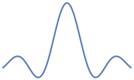
$$\int_0^\infty \frac{\varphi(x)}{x^x}\,dx = \sum_{k=-\infty}^{\infty} \frac{\varphi(k)}{k^k}. \tag{22.1}$$

It is doubtful that Ramanujan intended Entry 22 to be anything more than a proposed equality for which he probably tried to find examples. In Chapter 13 (Part II [4, pp. 226–227]), Ramanujan briefly considered a similar

Borwein's integrals

Consider the integrals

$$\int_{\mathbb{R}} \prod_{i=0}^{n} sinc\left(\frac{x}{2i+1}\right) dx, \qquad sinc\,(x) = \frac{\sin x}{x}$$



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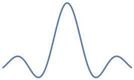

$$n = 0: \quad \int_{\mathbb{R}} sinc\,(x)\, dx = \pi$$

1. *Mr. Berry's first proof.** **This is that expressed by the series of equations**

$$\int_0^\infty \frac{\sin x}{x} dx = \int_0^\infty \lim_{a \to 0} \left(e^{-ax} \frac{\sin x}{x}\right) dx = \lim_{a \to 0} \int_0^\infty e^{-ax} \frac{\sin x}{x} dx$$

$$= \lim_{a \to 0} \int_0^\infty e^{-ax} dx \int_0^1 \cos tx\, dt = \lim_{a \to 0} \int_0^1 dt \int_0^\infty e^{-ax} \cos tx\, dx$$

$$= \lim_{a \to 0} \int_0^1 \frac{a\, dt}{a^2 + t^2} = \lim_{a \to 0} \arctan\left(\frac{1}{a}\right) = \tfrac{1}{2}\pi.$$

G.H. Hardy, the integral $\int_0^\infty \frac{\sin x}{x} dx$, Mathematical Gazette, 5, 98-103, 1909

Borwein's integrals

$$n = 1 : \quad \int_{\mathbb{R}} sinc\,(x)\, sinc\left(\frac{x}{3}\right) dx = \pi$$

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Borwein's integrals

$$\int_{\mathbb{R}} sinc\left(x\right) sinc\left(\frac{x}{3}\right) sinc\left(\frac{x}{5}\right) dx = \pi$$

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Borwein's integrals

$$n = 7: \quad \int_{\mathbb{R}} sinc\,(x)\, sinc\left(\frac{x}{3}\right) sinc\left(\frac{x}{5}\right) \ldots sinc\left(\frac{x}{13}\right) sinc\left(\frac{x}{15}\right) dx$$

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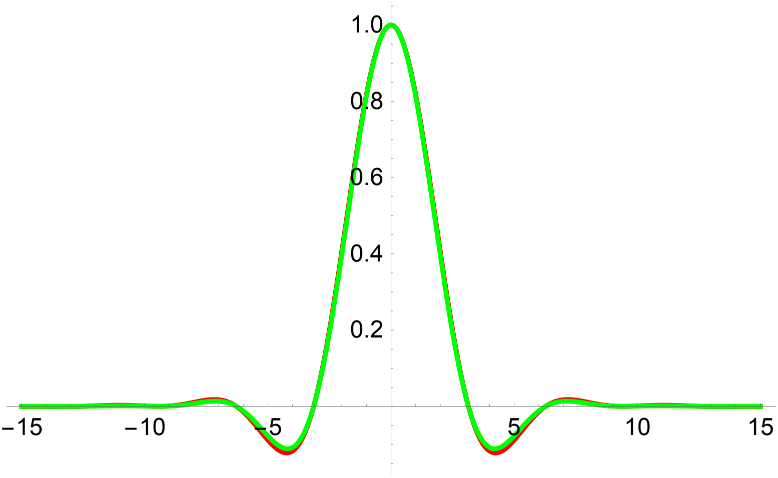
$$= \frac{467807924713440738696537864469}{467807924720320453655260875000}\pi$$

$$= \left(1 - 1.470628.10^{-11}\right)\pi$$

Borwein's integrals



Borwein's integrals: Ingredients

Fourier transform

$$\tilde{f}(z) = TF[f(x)](z) = \int_{\mathbb{R}} f(x)\, e^{-\imath 2\pi x z} dx$$

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with special case

$$sinc(z) = \pi TF\left[\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(x)\right](z) = \pi \int_{\mathbb{R}} \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(x) e^{-\imath 2\pi xz} dx$$

$$= \pi \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\imath 2\pi xz} dx$$

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Parseval identity

$$\int_{\mathbb{R}} \tilde{f}(z)\, \tilde{g}(z)\, dz = \int_{\mathbb{R}} f(x)\, g(x)\, dx$$

Borwein's integrals: More Ingredients

Dirac distribution

$$\delta\left(z\right) = \int_{\mathbb{R}} e^{\imath 2\pi x z} dx$$

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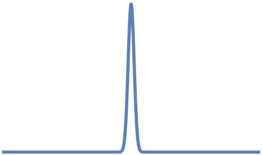

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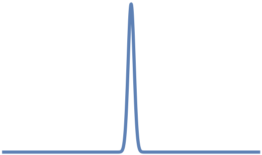

$$\delta(z) = \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} e^{\imath m z}, \quad z \in [-\pi, \pi]$$

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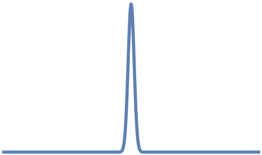

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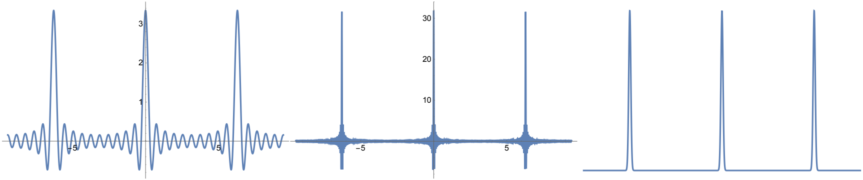

Borwein's integrals: the probabilistic approach

With $\{X_j\}_{1\le j\le m}$ independent random variables, with characteristic functions $\psi_j(x) = \mathbb{E}e^{\imath 2\pi X_j}$,

$$I = \int_{\mathbb{R}} sinc(a_0 x) \prod_{j=1}^{n} \psi_j(a_j x)\, dx$$

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Using Parseval identity with $Z=\sum_{j=1}^{m} a_j X_j$,

$$I=\pi\int_{\mathbb{R}}\mathbb{1}_{\left[-\frac{a_0}{2},\frac{a_0}{2}\right]}\left(z\right)f_Z\left(z\right)dz$$

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Assuming each X_j has bounded support $\left[-\frac{1}{2}, \frac{1}{2}\right]$, and $\sum_{j=1}^{n} a_j < a_0$,

$$I = \pi \int_{\mathbb{R}} f_Z(z)\, dz = \pi$$

Borwein's integrals

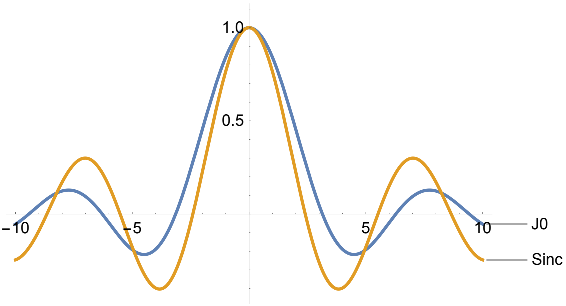
In Borwein's case

$$a_i = \frac{1}{2i+1}$$

so that

$$\sum_{i=1}^{6} a_i = \frac{43024}{45045} < a_0 = 1, \quad \sum_{i=1}^{7} a_i = \frac{46027}{45045} > a_0 = 1$$

Borwein's integrals: generalization



Borwein's integrals: generalization

The proof suggests this result holds for any random variable X_j with bounded support $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Take for example

$$f_{X_j}(x) = \begin{cases} \frac{1}{\pi}\frac{1}{\sqrt{1-4x^2}}, & -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 0, & \text{else} \end{cases}$$

so that

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Borwein's integrals: geometric interpretation

Borwein's integral is a **probability**: with U_i uniform over $[-1, 1]$,

$$\frac{1}{\pi}\int_{\mathbb{R}}\prod_{i=0}^{n} sinc\left(a_i x\right) dx = \Pr\left\{\left|a_1 U_1 + \ldots a_n U_n\right| \leq a_0\right\}$$

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As long as

$$\sum_{i=1}^{n} a_i \leq a_0$$

$$\Pr\left\{\left|a_1 U_1 + \ldots a_n U_n\right| \leq a_0\right\} = 1.$$

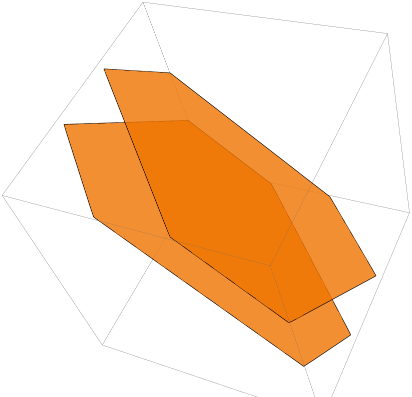
Borwein's integrals: geometric interpretation

Borwein's integral is a **volume,** a result due to Polya (1912)

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(\theta x)}{x}\prod_{i=1}^{n} sinc\,(w_i x)\,dx = Vol\left([-1,1]^n \cap S_{\mathrm{w},\theta}\right)$$

with

$$S_{w,\theta} = \left\{\mathrm{x}\in\mathbb{R}^n : |\mathrm{w.x}| \le \frac{\theta}{2}\right\}$$



Borwein's integrals: sums are integrals

Another remarkable identity: assuming $k_j \geq 0$ and $k_1 + \cdots + k_m < 2\pi$,

$$\int_{\mathbb{R}} \prod_{j=1}^{n} sinc\,(k_j x)\, dx = \sum_{m=-\infty}^{\infty} \prod_{j=1}^{m} sinc\,(k_j m)$$

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Proof:

$$\int_{\mathbb{R}} \prod_{j=1}^{n} sinc\left(k_j x\right) dx = \int_{\mathbb{R}} \mathbb{E} e^{\imath x \sum_j k_j U_j} dx$$

$$= \mathbb{E} \int_{\mathbb{R}} e^{\imath x \sum_j k_j U_j} dx = \mathbb{E}\delta\left(\frac{Z}{2\pi}\right)$$

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with

$$Z = \sum_{j=1}^{m} k_j U_j$$

and finally

$$\mathbb{E}\delta\left(\frac{Z}{2\pi}\right) = \int f_Z\left(z\right) \delta\left(\frac{z}{2\pi}\right) dz = 2\pi f_Z\left(0\right)$$

Bernoulli's sophomore's dream

In 1697, Bernoulli [2] found that

$$\int_0^1 x^x dx = \sum_{k\geq 1} \frac{(-1)^k}{k^k} = 0.783431...$$

[2] Johan Bernoulli, Demonstratio Methodi Analyticae, qua usus est pro determinanda aliqua Quadratura exponentiali per seriem, Actis Eruditorum A (1697), p. 131.

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Alternatively

$$\int_0^1 \frac{1}{x^x} dx = \sum_{k\geq 1} \frac{1}{k^k} = 1.29128...$$

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Bernoulli's sophomore's dream

Not difficult to prove:

$$\int_0^1 \frac{1}{x^x} dx = \int_0^1 e^{-x \log x} dx = \int_0^1 \sum_{n \geq 0} \frac{(-1)^n}{n!} x^n \log^n x dx$$

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$$= \sum_{n \geq 0} \frac{(-1)^n}{n!} \int_0^1 x^n \log^n x dx$$

with

$$I_n = \int_0^1 x^n \log^n x dx = \frac{(-1)^n n!}{(n+1)^{n+1}}$$

Bernoulli's sophomore's dream

Not difficult to prove:

$$\int_0^1 \frac{1}{x^x} dx = \int_0^1 e^{-x \log x} dx = \int_0^1 \sum_{n \geq 0} \frac{(-1)^n}{n!} x^n \log^n x dx$$

$$= \sum_{n \geq 0} \frac{(-1)^n}{n!} \int_0^1 x^n \log^n x dx$$

with

$$I_n = \int_0^1 x^n \log^n x dx = \frac{(-1)^n \, n!}{(n+1)^{n+1}}$$

so that

$$\int_0^1 \frac{1}{x^x} dx = \sum_{n \geq 0} \frac{1}{(n+1)^{n+1}} = \sum_{n \geq 1} \frac{1}{n^n}$$

Bernoulli's sophomore's dream

Compute I_n considering $K_n(\alpha = 0)$ with

$$K_n(\alpha) = \int_0^1 x^{n+\alpha} \log^n x dx = \frac{d^n}{d\alpha^n} \int_0^1 x^{n+\alpha} dx$$

$$= \frac{d^n}{d\alpha^n} \frac{1}{n+\alpha+1} = \frac{(-1)^n n!}{(n+\alpha+1)^{n+1}}$$

Bernoulli's sophomore's dream

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A073009 Decimal expansion of Sum_{n >= 1} 1/n^n. 48

1, 2, 9, 1, 2, 8, 5, 9, 9, 7, 0, 6, 2, 6, 6, 3, 5, 4, 0, 4, 0, 7, 2, 8, 2, 5, 9, 0, 5, 9, 5, 6, 0, 0, 5, 4, 1, 4, 9, 8, 6, 1, 9, 3, 6, 8, 2, 7, 4, 5, 2, 2, 3, 1, 7, 3, 1, 0, 0, 0, 2, 4, 4, 5, 1, 3, 6, 9, 4, 4, 5, 3, 8, 7, 6, 5, 2, 3, 4, 4, 5, 5, 5, 5, 8, 8, 1, 7, 0, 4, 1, 1, 2, 9, 4, 2, 9, 7, 0, 8, 9, 8, 4, 9, 9 (list; constant; graph; refs; listen; history; text; internal format)

OFFSET 1,2

LINKS
Kenny Lau, Table of n, a(n) for n = 1..10001
Johan Bernoulli, Demonstratio Methodi Analyticae, qua usus est pro determinanda aliqua Quadratura exponentiali per seriem, Actis Eruditorum A (1697), p. 131. Collected in Opera Omnia, vol. 3, 1742. See p. 376ff.
M. L. Glasser, A note on Beukers's and related integrals, Amer. Math. Monthly 126(4) (2019), 361-363.
Jaroslav Hančl and Simon Kristensen, Metrical irrationality results related to values of the Riemann zeta-function, arXiv:1802.03946 [math.NT], 2018.
Randall Munroe, Approximations, xkcd Web Comic #1047, Apr 25 2012.
Simon Plouffe, Sum(1/n^n, n=1..infinity). [internet archive]
Eric Weisstein's World of Mathematics, Power Tower.
Eric Weisstein's World of Mathematics, Sophomore's Dream.

FORMULA
Equals Integral_{x = 0..1} dx/x^x.
Constant also equals the double integral Integral_{y = 0..1} Integral_{x = 0..1} 1/(x*y)^(x*y) dx dy. - Peter Bala, Mar 04 2012
Approximately log(3)^e, see Munroe link. - Charles R Greathouse IV, Apr 25 2012

Bernoulli's sophomore's dream

It was suggested (OEIS A073009) that

$$\sum_{k\geq 1} k^{-k} = \int_0^1 x^{-x} dx = \iint_{[0,1]^2} (xy)^{-xy}\, dxdy$$

Bernoulli's sophomore's dream

It was suggested (OEIS A073009) that

$$\sum_{k\geq 1} k^{-k} = \int_0^1 x^{-x} dx = \iint_{[0,1]^2} (xy)^{-xy}\, dxdy$$

Easy proof:

$$\iint_{[0,1]^2} (xy)^{-xy}\, dxdy = \sum_{k,l\geq 1} \frac{1}{(k+l-1)^{k+l}} = \sum_{k\geq 1} \frac{1}{k^k}$$

because

$$\sum_{k,l\geq 1} \frac{1}{(k+l-1)^{k+l}} = \sum_{p\geq 2} \frac{p-1}{(p-1)^p} = \sum_{p\geq 1} \frac{1}{p^p}$$

Bernoulli's sophomore's dream

Another way: use Glasser's theorem [3] [4]

$$\iint_{[0,1]^2} f(xy)\, dxdy = -\int_0^1 f(x) \log x dx$$

[3] M. L. Glasser, A Note on Beukers's and Related Double Integrals, The American Mathematical Monthly Volume 126, 2019 - Issue 4

[4] a special case of $\int_{[0,1]^n} f(x_1 \ldots x_n)\, dx_1 \ldots dx_n = \int_0^1 f(x) \frac{(-\log x)^{n-1}}{(n-1)!} dx$.

Bernoulli's sophomore's dream

Another way: use Glasser's theorem [3] [4]

$$\iint_{[0,1]^2} f(xy)\,dxdy = -\int_0^1 f(x)\log x dx$$

With $f(x) = x^{-x}$

$$\iint_{[0,1]^2} f(xy)\,dxdy = -\int_0^1 x^{-x}\log x dx$$

and since

$$\frac{d}{dx}x^{-x} = x^{-x}(-1-\log x),$$

$$\iint_{[0,1]^2} (xy)^{-xy}\,dxdy = -\int_0^1 x^{-x}\log x dx = \int_0^1 x^{-x}dx + \overbrace{\int_0^1 \frac{d}{dx}x^{-x}dx}^{0}$$

[3] M. L. Glasser, A Note on Beukers's and Related Double Integrals, The American Mathematical Monthly Volume 126, 2019 - Issue 4

[4] a special case of $\int_{[0,1]^n} f(x_1 \dots x_n)\,dx_1 \dots dx_n = \int_0^1 f(x)\frac{(-\log x)^{n-1}}{(n-1)!}dx$.

Bernoulli's sophomore's dream: Directions of research

- extend

$$\iint_{[0,1]^2} (xy)^{-xy}\, dxdy = \int_0^1 x^{-x} dx = \sum_{k\geq 1} \frac{1}{k^k}$$

to

$$\iint_{[0,1]^p} (x_1 \ldots x_p)^{-x_1 \ldots x_p}\, dx_1 \ldots dx_p = \sum_{k\geq 1} \frac{\binom{k+p-2}{p-1}}{k^{k+p-1}}$$

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$$\iint_{[0,1]^2} x^{-xy} dxdy = \sum_{k\geq 1} \frac{1}{k^{k+1}}$$

to

$$\iint_{[0,1]^p} x_1^{-x_1 \dots x_p} dx_1 \dots dx_p = \sum_{k\geq 1} \frac{1}{k^{k+p-1}}$$

Bernoulli's sophomore's dream: Directions of research

- study the properties of

$$\zeta(p) = \sum_{k \geq 1} \frac{1}{k^{k+p-1}}$$

as compared to those of the Riemann zeta function

$$\zeta_R(p) = \sum_{k \geq 1} \frac{1}{k^p}$$

Bernoulli's sophomore's dream: Directions of research

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$$\zeta_R(p) = \sum_{k\geq 1} \frac{1}{k^p}$$

- prove

$$\lim_{p\to\infty} \zeta(p) = 1$$

Bernoulli's sophomore's dream: Directions of research

- what is the equivalent of

$$\sum_{p\geq 2}\left(\zeta_R\left(p\right)-1\right)=1$$

[5] H.M. Srivastava and J. Choi, zeta and q-zeta functions and associated series and integrals, 2012, Elsevier

Bernoulli's sophomore's dream: Directions of research

- what is the equivalent of

$$\sum_{p\geq 2}\left(\zeta_R\left(p\right)-1\right)=1$$

- what is the equivalent of

$$\left(n+\frac{1}{2}\right)\zeta_R\left(2n\right)=\sum_{m=1}^{n-1}\zeta_R\left(2m\right)\zeta_R\left(2n-2m\right)$$

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- or of one of the 464 identities in [5]

---

[5] H.M. Srivastava and J. Choi, zeta and q-zeta functions and associated series and integrals, 2012, Elsevier

Bernoulli's sophomore's dream: Directions of research

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)-1}{2^{2k}} = \frac{1}{6}; \tag{211}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)-1}{2^{2k}} = -\frac{4}{3} + 2\log 2; \tag{212}$$

$$\sum_{k=2}^{\infty} (-1)^k \{\zeta(k)-1\} = \frac{1}{2}; \tag{213}$$

$$\sum_{k=2}^{\infty} \{\zeta(k)-1\} = 1; \tag{214}$$

$$\sum_{k=1}^{\infty} \{\zeta(2k)-1\} = \frac{3}{4}; \tag{215}$$

$$\sum_{k=1}^{\infty} \{\zeta(2k+1)-1\} = \frac{1}{4}; \tag{216}$$

$$\sum_{k=2}^{\infty} (-1)^k \{\zeta(k)-1\} \left(\frac{3}{2}\right)^k = \frac{31}{10} - 3\log 2; \tag{217}$$

Bernoulli's sophomore's dream: Directions of research

- an equivalent of the Multiple Zeta Value

  $$\zeta_2(m,n) = \sum_{p,q\geq 0} \frac{1}{(p+1)^m (p+q+1)^n} = \sum_{q\geq p\geq 0} \frac{1}{(p+1)^m (q+1)^n}$$

  would be

  $$\sum_{p,q\geq 0} \frac{1}{(p+1)^{p+1} (p+q+1)^{q+1}} = \iint_{[0,1]^2} x^{-xy} y^{-y} dxdy$$

Bernoulli's sophomore's dream: Directions of research

- a hand to hand combat: PSLQ

Bernoulli's sophomore's dream: Directions of research

- a hand to hand combat: PSLQ
- Wolfram Alpha:

  $$\text{Identify[3.141592653]}$$

  returns

  $$Pi$$

Bernoulli's sophomore's dream: Directions of research

- a hand to hand combat: PSLQ
- Wolfram Alpha:

  Identify[3.141592653]

  returns

  $$Pi$$

- Mathematica:

  FindIntegerNullVector[{Log[2], Log[4]]})

  returns

  $$\{-2, 1\}$$

Bernoulli's sophomore's dream: Directions of research

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ELSEVIER

Hand-to-hand combat with thousand-digit integrals

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ABSTRACT

In this paper we describe numerical investigations of definite integrals that arise by considering the moments of multi-step uniform random walks in the plane, together with a closely related class of integrals involving the elliptic functions K, K', E and E'. We find that in many cases such integrals can be "experimentally" evaluated in closed form or that intriguing linear relations exist within a class of similar integrals. Discovering these identities and relations often requires the evaluation of integrals to extreme precision, combined with large-scale runs of the "PSLQ" integer relation algorithm. This paper presents details of the techniques used in these calculations and mentions some of the many difficulties that can arise.

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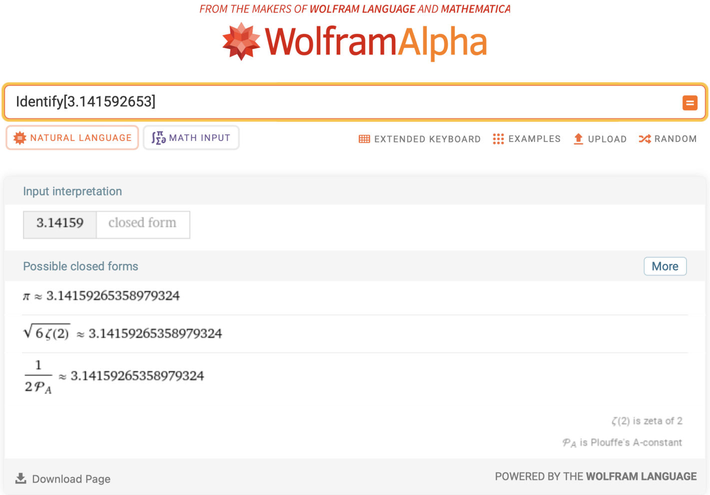
Bernoulli's sophomore's dream: Directions of research



Bernoulli's sophomore's dream: Directions of research

$$\zeta(1) = \sum_{k \geq 1} \frac{1}{k^k} = 1.291285997062663540\underline{407}\ldots$$

$$\frac{1660 + 550\sqrt{\pi} - 15\pi - 16\pi\sqrt{\pi} + 148\pi^2}{976\pi}$$

$$= 1.291285997062663540\underline{399}\ldots$$

Bernoulli's sophomore's dream: Directions of research

$$\zeta(1) = \sum_{k\geq 1} \frac{1}{k^k} = 1.2912859970626635\underline{407}\ldots$$

$$\frac{1660 + 550\sqrt{\pi} - 15\pi - 16\pi\sqrt{\pi} + 148\pi^2}{976\pi}$$

$$= 1.2912859970626635\underline{399}\ldots$$

$$\frac{\zeta(3)}{\pi^2} = \frac{1}{\pi^2}\sum_{k\geq 1}\frac{1}{k^{k+2}} = 0.10809681368424696\underline{40436}$$

$$-\frac{5\left(16 - 315\pi + 98\pi^2\right)}{1116 - 277\pi + 5\pi^2} = 0.10809681368424696\underline{36049}$$

A bonus identity

the sum of digits function: with the base 2 expansion

$$n = n_k n_{k-1} \dots n_0, \qquad n_j \in \{0, 1\},$$

meaning

$$n = \sum_{j=0}^{k} 2^j n_j,$$

define

$$s_2(n) = \sum_{j=0}^{k} n_j.$$

A bonus identity

For example,

$$s_2(9) = s_2(1001) = 2.$$

We have the three identities

$$\prod_{n\geq 1}\left(\frac{1+\frac{1}{n}}{1+\frac{1}{n+1}}\frac{1+\frac{1}{2n+2}}{1+\frac{1}{2n}}\right) = \frac{3}{4}$$

$$\prod_{n\geq 1}\left(\frac{1+\frac{1}{n}}{1+\frac{1}{n+1}}\frac{1+\frac{1}{2n+2}}{1+\frac{1}{2n}}\right)^{s_2(n)} = \frac{\pi}{2}$$

$$\prod_{n\geq 1}\left(\frac{1+\frac{1}{2n}}{1+\frac{1}{2n+2}}\frac{1+\frac{1}{4n+4}}{1+\frac{1}{4n}}\right)^{s_2(n)} = \frac{\pi}{2}\prod_{k=1}^{\infty}\tanh^2\left(k\frac{\pi}{2}\right) = 2\sqrt{\frac{2}{\pi}}\Gamma^2\left(\frac{5}{4}\right)$$

A bonus identity

Mathematica evaluates

$$\prod_{k=1}^{\infty} \tanh\left(k\frac{\pi}{2}\right) = 0.91357913815611682141\ldots$$

numerically only. However with $q = e^{-\pi}$,

$$\prod_{k=1}^{\infty} \tanh\left(k\frac{\pi}{2}\right) = \theta_4\left(0, e^{-\pi}\right) = \left(q^2, q^2\right)_\infty \left(q, q^2\right)_\infty^2 .$$

Jacobi theta functions

$$\theta_4(q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}$$

$$\theta_2(q) = \sum_{n\in\mathbb{Z}} q^{\left(n+\frac{1}{2}\right)^2}$$

with the factorization (Jacobi triple product identity)

$$\theta_4(q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} = \prod_{m\geq 1} \left(1 - q^{2m}\right)\left(1 - q^{2m-1}\right)^2$$

$$= \left(q^2; q^2\right)_\infty \left(q; q^2\right)_\infty^2$$

A bonus identity

Proof: with $q = e^{-\frac{\pi}{2}}$

$$\prod_{k=1}^{\infty} \tanh\left(k\frac{\pi}{2}\right) = \prod_{k=1}^{\infty} \frac{1-q^k}{1+q^k}$$

and since

$$\prod_{k=1}^{\infty} \frac{1}{1+q^k} = \prod_{k=0}^{\infty} 1 - q^{2k+1},$$

we deduce

$$\prod_{k=1}^{\infty} \tanh\left(k\frac{\pi}{2}\right) = \prod_{k=0}^{\infty} \left(1-q^{k+1}\right)\left(1-q^{2k+1}\right)$$

$$= \prod_{k=0}^{\infty} \left(1-q^{2k+1}\right)\left(1-q^{2k+2}\right)\left(1-q^{2k+1}\right) = \left(q^2, q^2\right)_{\infty} \left(q, q^2\right)_{\infty}^2$$

Approximating $\prod_{k=1}^{\infty} \tanh\left(k\frac{\pi}{2}\right) = 0.91357913815611682$

This produces the series representation

$$\prod_{k=1}^{\infty} \tanh\left(k\frac{\pi}{2}\right) = \sum_{n\in\mathbb{Z}} (-1)^n e^{-\pi n^2} = \theta_4\left(e^{-\pi}\right)$$

and the approximation [6]

$$1 - 2e^{-\pi} + 2e^{-4\pi} = 0.913579138157\underline{16791844}$$

By modularity, we have

$$\theta_4\left(e^{-\pi}\right) = \sum_{n\in\mathbb{Z}} (-1)^n e^{-\pi n^2} = \sum_{n\in\mathbb{Z}} e^{-\pi\left(n+\frac{1}{2}\right)^2} = \theta_2\left(e^{-\pi}\right)$$

with the approximation

$$e^{-\pi\left(\frac{3}{2}\right)^2} + e^{-\pi\left(-\frac{3}{2}\right)^2} + e^{-\pi\left(\frac{1}{2}\right)^2} + e^{-\pi\left(-\frac{1}{2}\right)^2} = 4e^{-\pi\frac{5}{4}} \cosh \pi$$

$$= 0.91357913\underline{221760278960}$$

---

[6] $\prod_{k=1}^{5} \tanh\left(k\frac{\pi}{2}\right) = 0.91357915\underline{059276074}$