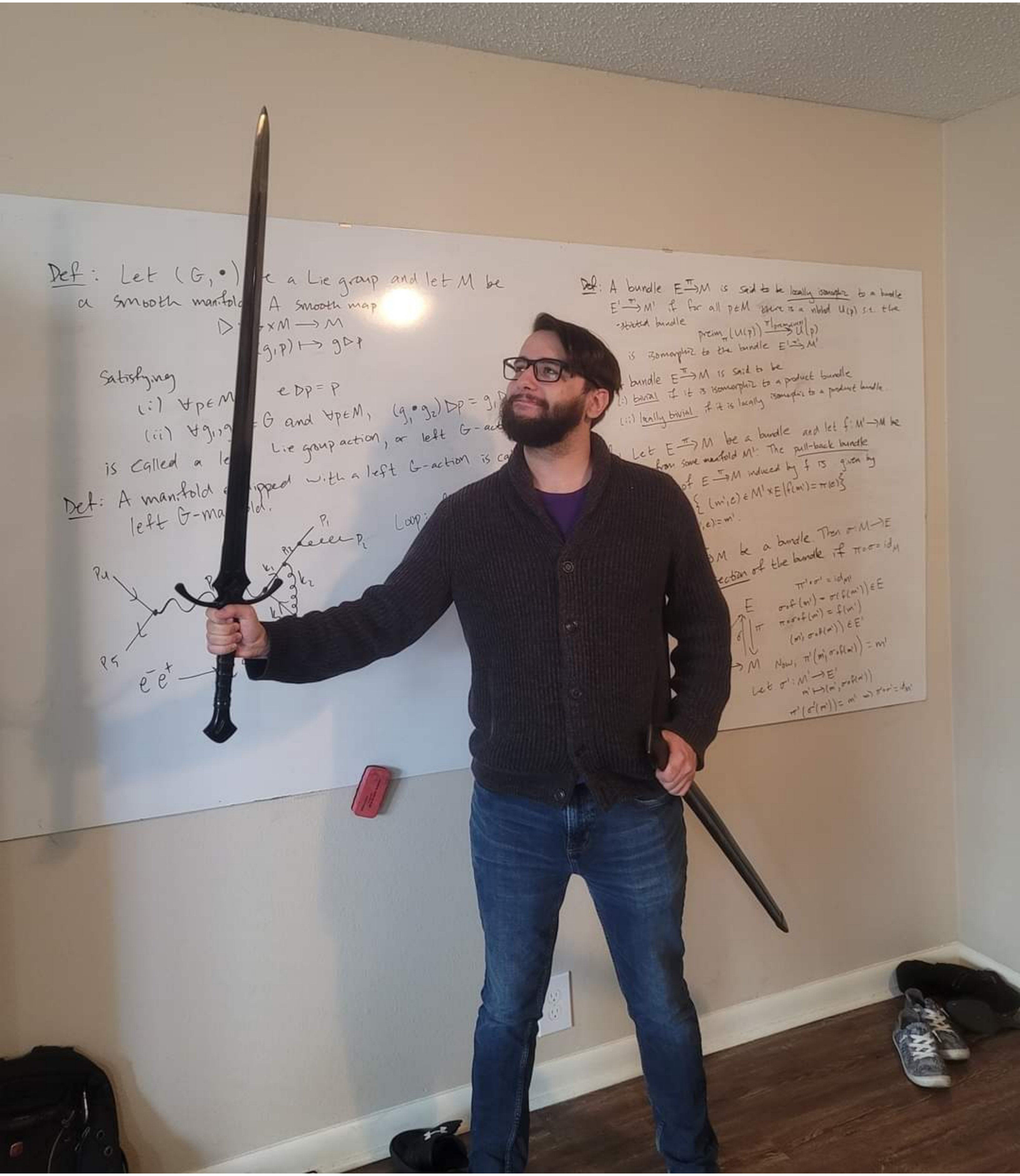
Dubious Identities:

a Visit to the Borwein Zoo

C. Vignat, Tulane, Orsay

Z. Bradshaw, Tulane



Def: Let (G,•)
a Lie group and let M be
a smooth
A smooth map
Satisfying
is called a
Lie group action, or left G-act
Def: A manifold
with a left G-action is ca
left G-ma
Loop:
$e^- e^+$
Def: A bundle $E \xrightarrow{\pi} M$ is said to be locally isomorphic to a bundle $E' \xrightarrow{\pi'} M'$ if for all $p \in M$
is isomorphic to the bundle $E' \xrightarrow{\pi'} M'$
bundle $E \xrightarrow{\pi} M$ is said to be
trivial if it is isomorphic to a product bundle
locally trivial if it is locally isomorphic to a product bundle
Let $E \xrightarrow{\pi} M$ be a bundle and let $f: M' \to M$ be
The pull-back bundle
of $E \xrightarrow{\pi} M$ induced by f is given by
M be a bundle. Then
section of the bundle
Let $\sigma': M' \to E'$

Strange Series and High Precision Fraud

J. M. Borwein; P. B. Borwein

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Strange Series and High Precision Fraud

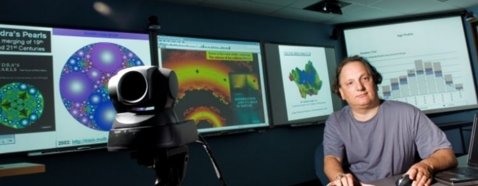
J. M. Borwein and P. B. Borwein

INTRODUCTION. Five of the following twelve series approximations are exact. The remaining seven are not identities but are approximations that are correct to at least 30 digits. One in fact is correct to over 18,000 digits and another to in excess of a billion digits. The reader is invited to separate the true from the bogus. (For answers see the end of the introduction.) Most of these series are easily amenable to high precision calculation in one's favorite high precision environment, such as Maple or MACSYMA, and provide examples of "caveat computat." Things are not always as they appear.

Sum 1

$$\sum_{n=1}^{\infty} \frac{a(2^n)}{2^n} \doteq \frac{1}{99}$$

where $a(n)$ counts the number of odd digits in odd places in the decimal expansion of n. ($a(901) = 2$, $a(210) = 0$, $a(811) = 1$, here the 1st digit is the 1st to the left of the decimal point.)



dra's Pearls
merging of 19th
d 21st Centuries
2002: http://klein.math
Age Profile

Borwein's Gaussian identity

$$\frac{1}{10^5} \sum_{n \in \mathbb{Z}} \exp\left(-\frac{n^2}{10^{10}}\right) = \sqrt{\pi}$$

Is it true ?

Answer : it is wrong but

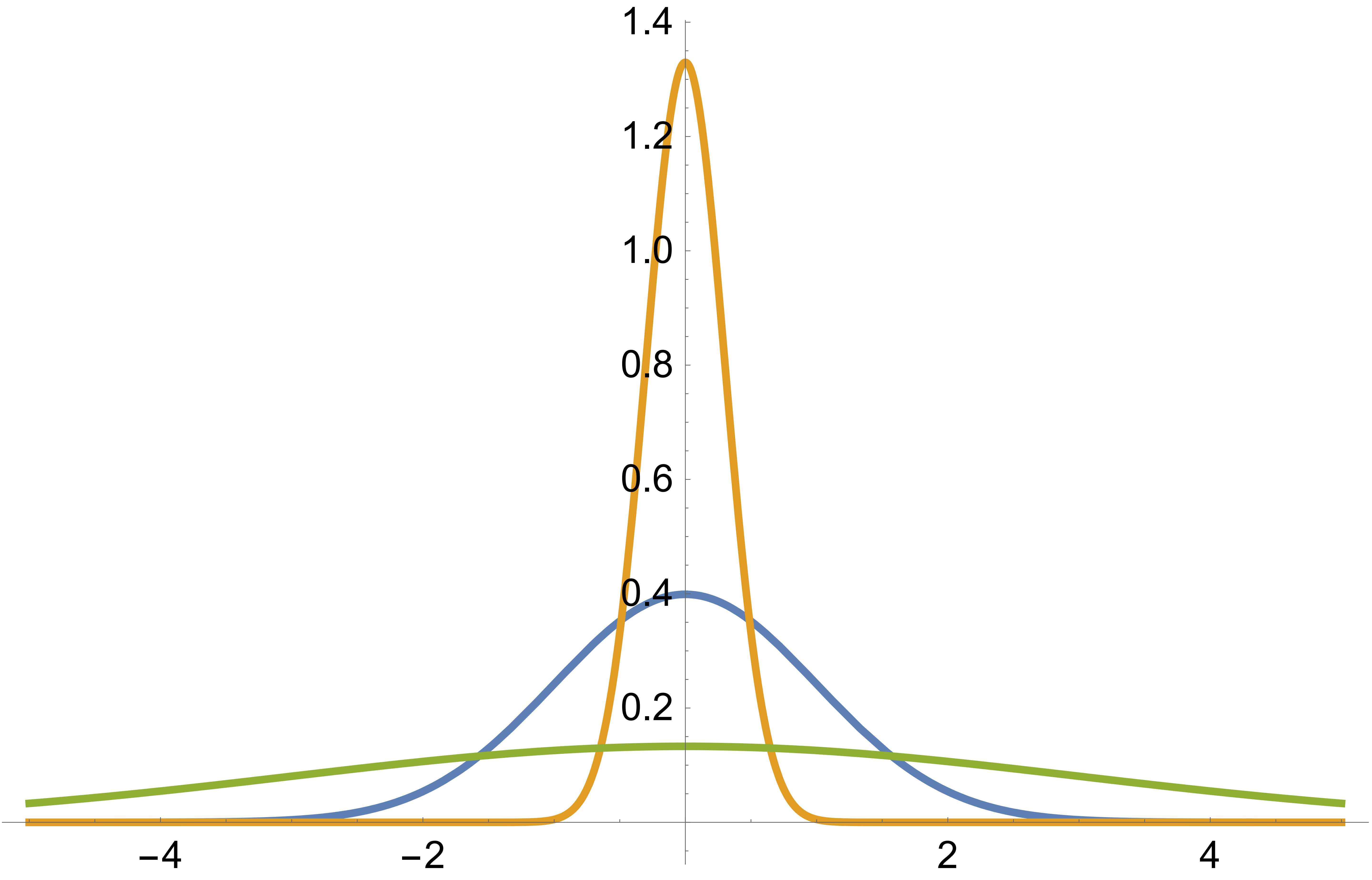
$$\left| \sqrt{\pi} - \frac{1}{10^5} \sum_{n \in \mathbb{Z}} \exp\left(- \frac{n^2}{10^{10}} \right) \right|$$

$$\leq 10^{-42000000000}$$

Take a Gaussian (continuous) distribution with variance σ^2:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

- mean $= 0$
- variance $= \sigma^2$

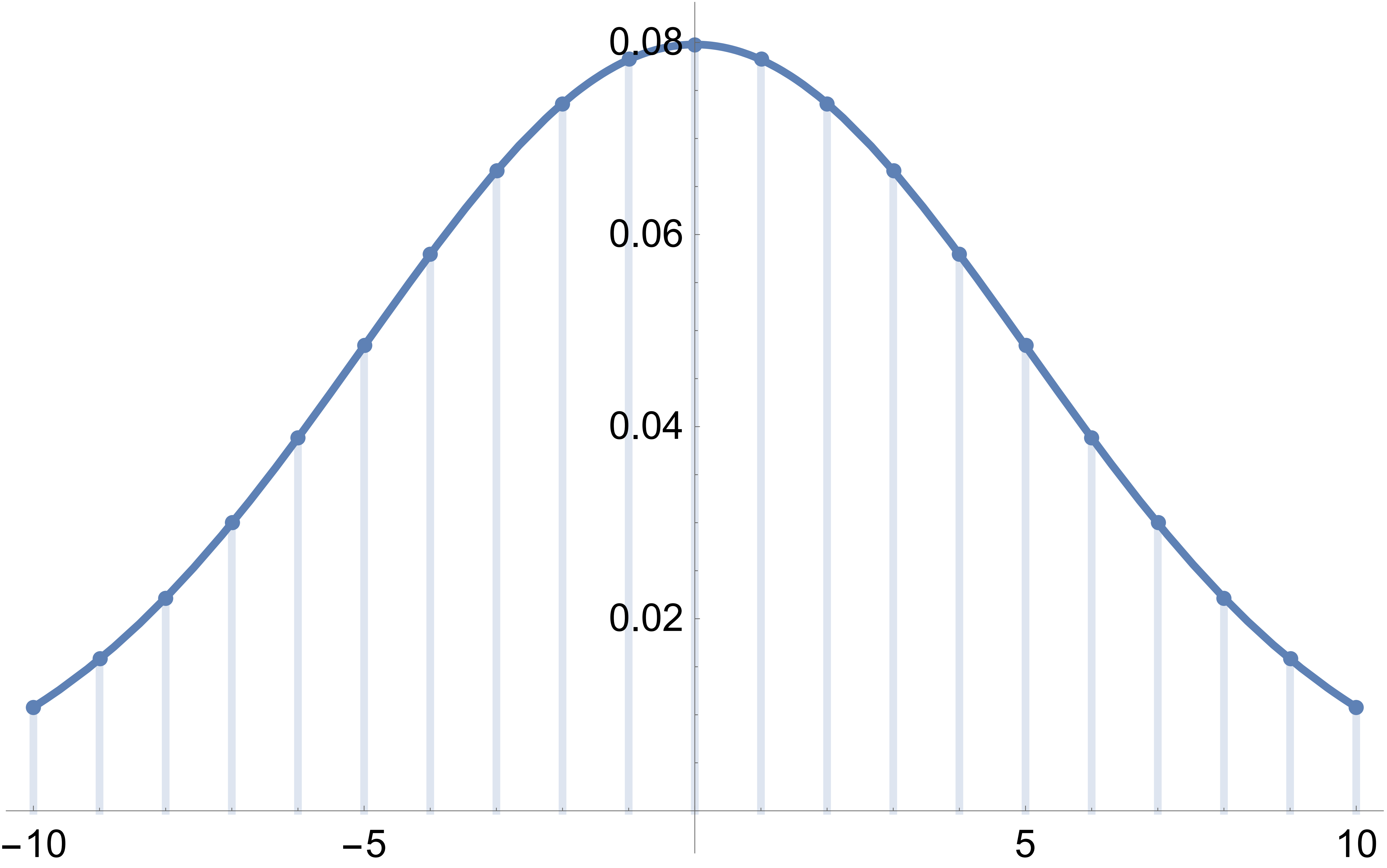


1.4
1.2
1.0
0.8
0.6
0.4
0.2
−4
−2
2
4

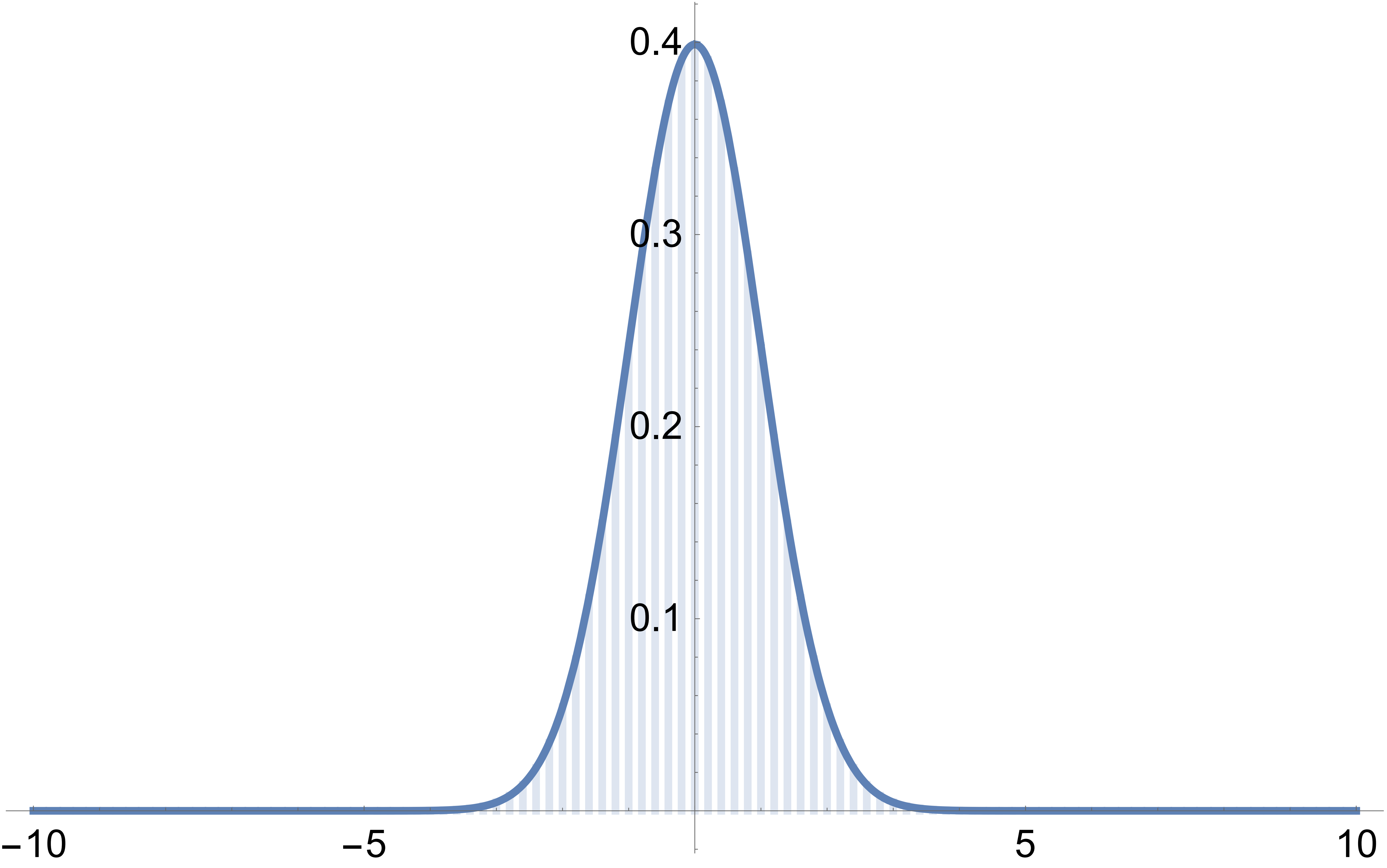
A scaling effect: take $f(x)$ any probability measure, $a > 0$

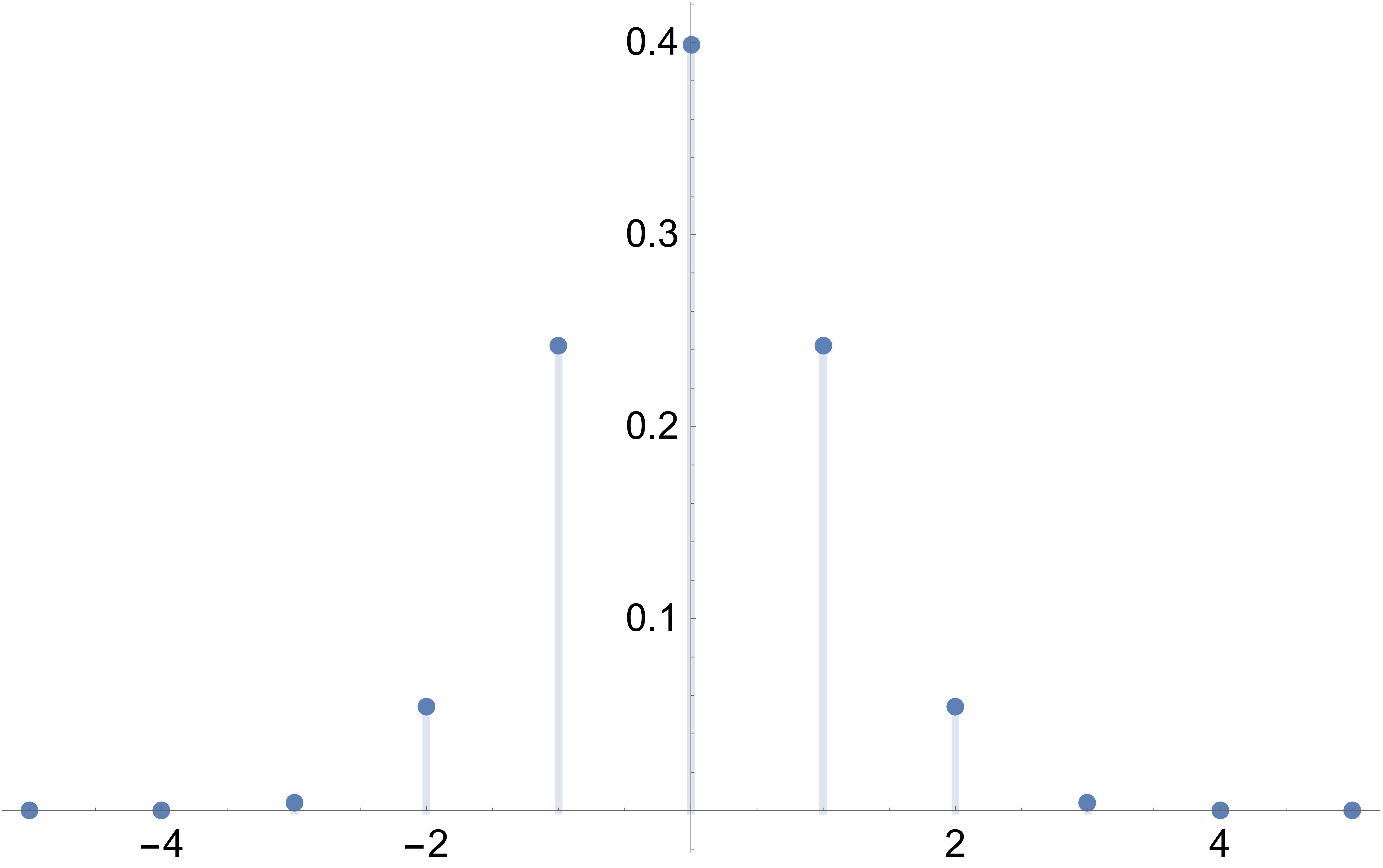
$$\int_{\mathbb{R}} f(x)\,dx = 1 \quad \Rightarrow \frac{1}{a}\int_{\mathbb{R}} f\left(\frac{x}{a}\right) dx = 1$$

$$\frac{1}{a}\int_{-\infty}^{\infty} f\left(\frac{x}{a}\right) dx \simeq \frac{1}{a}\sum_{k\in\mathbb{Z}} f\left(\frac{k}{a}\right)$$



0.08
0.06
0.04
0.02
−10
−5
5
10


0.4
0.3
0.2
0.1
−10
−5
5
10

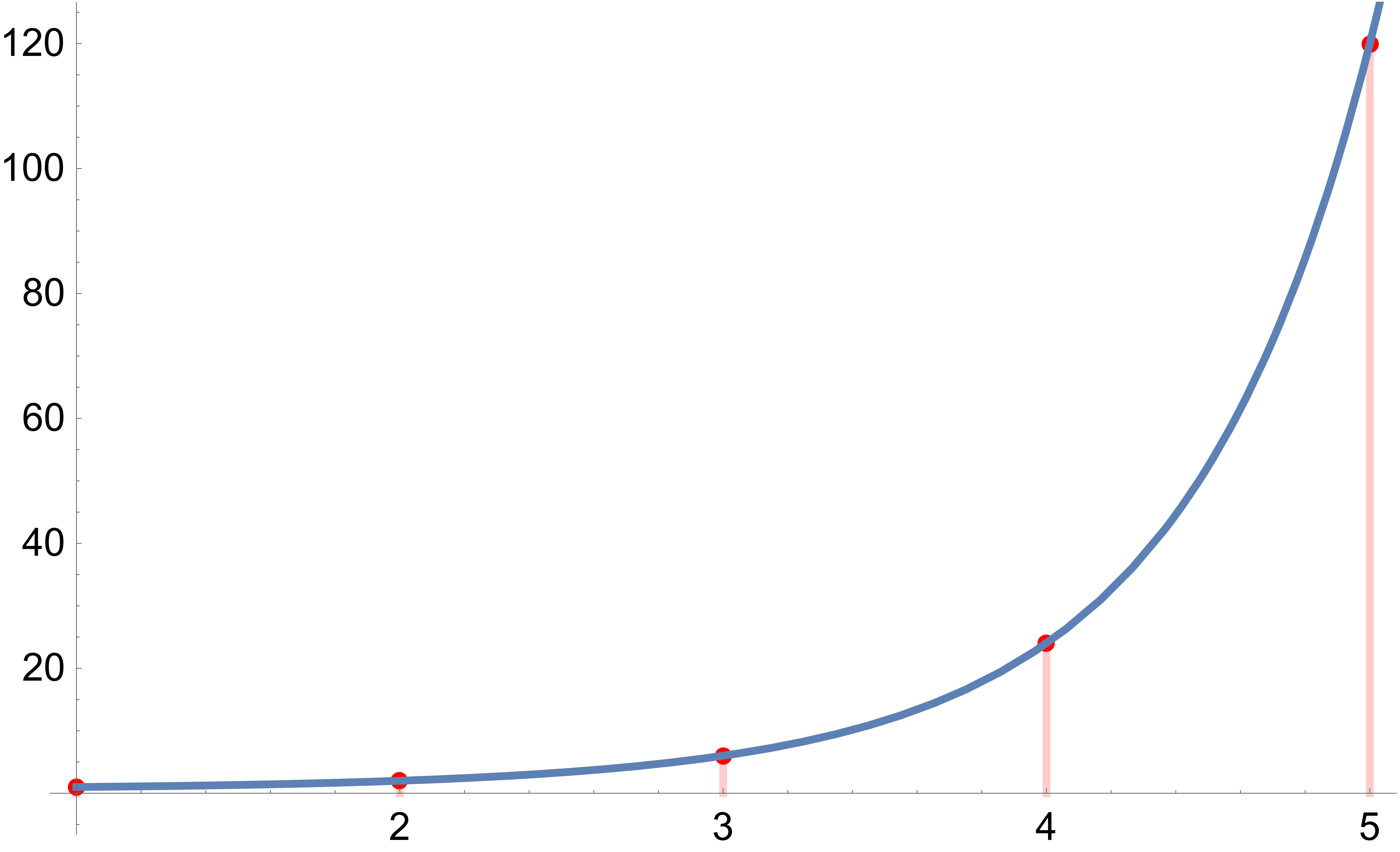

0.4
0.3
0.2
0.1
−4
−2
2
4

For example : if $\lambda > \frac{1}{2}$

$$\frac{1}{10^5} \sum_{n \in \mathbb{Z}} \left(1 + \frac{n^2}{10^{10}}\right)^{-\lambda} \simeq \frac{\Gamma(\frac{1}{2})\Gamma(\lambda - \frac{1}{2})}{\Gamma(\lambda)}$$

with a maximal error $= 10^{-136440}$

for $\lambda = \pi . 10^5$


120
100
80
60
40
20
2
3
4
5

Infinite Product of Matrices

W. Gosper

$$\prod_{k=1}^{\infty} \begin{bmatrix} -\frac{k}{2(2k+1)} & \frac{5}{4k^2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \zeta(3) \\ 0 & 1 \end{bmatrix}$$

with Rieman's zeta function

$$\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}$$

Extension to the $(N+1)\times(N+1)$ case

$$
\prod_{k=1}^{\infty}
\begin{bmatrix}
\frac{-k}{2(2k+1)} & \frac{1}{2k(2k+1)} & 0 & \cdots & 0 & \frac{1}{k^{2N}} \\
0 & \frac{-k}{2(2k+1)} & \frac{1}{2k(2k+1)} & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots \\
\vdots & & \ddots & \ddots & \frac{1}{2k(2k+1)} & \frac{1}{k^4} \\
\vdots & & & \ddots & \frac{-k}{2(2k+1)} & \frac{5}{4k^2} \\
0 & \cdots & \cdots & \cdots & 0 & 1
\end{bmatrix}
=
\begin{bmatrix}
0 & \cdots & 0 & \zeta(2N+1) \\
\vdots & & \vdots & \zeta(2N-1) \\
\vdots & & \vdots & \vdots \\
\vdots & & \vdots & \zeta(3) \\
0 & \cdots & 0 & 1
\end{bmatrix}
$$

The even case (3×3)

$$\prod_{n \geq 1} \begin{bmatrix} \frac{n}{2(2n+1)} & \frac{-1}{2n(2n+1)} & \frac{1}{2n^3} \\ 0 & \frac{n}{2(2n+1)} & \frac{3}{2n} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \zeta(4) \\ 0 & 0 & \zeta(2) \\ 0 & 0 & 1 \end{bmatrix}$$

is true

The even case (4×4)

$$\prod_{n\geq 1}\begin{bmatrix} \frac{n}{2(2n+1)} & \frac{-1}{2n(2n+1)} & 0 & \frac{1}{2n^5} \\ 0 & \frac{n}{2(2n+1)} & \frac{-1}{2n(2n+1)} & \frac{1}{2n^3} \\ 0 & 0 & \frac{n}{2(2n+1)} & \frac{1}{2n} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \zeta(6) \\ 0 & 0 & 0 & \zeta(4) \\ 0 & 0 & 0 & \zeta(2) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is wrong

$$\prod_{n\geq 1}\begin{bmatrix} \frac{n}{2(2n+1)} & \frac{-1}{2n(2n+1)} & 0 & \frac{1}{2n^5} - \frac{9}{2n}H^{(4)}_{n-1} \\ 0 & \frac{n}{2(2n+1)} & \frac{-1}{2n(2n+1)} & \frac{1}{2n^3} \\ 0 & 0 & \frac{n}{2(2n+1)} & \frac{1}{2n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \zeta(6) \\ 0 & 0 & 0 & \zeta(4) \\ 0 & 0 & 0 & \zeta(2) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H^{(4)}_n = \sum_{k=1}^{n} \frac{1}{k^4}$$

$$\prod_{n\geq 1}\begin{bmatrix} \frac{n}{2(2n+1)} & \frac{-1}{2n(2n+1)} & 0 & \frac{1}{2n^5} \\ 0 & \frac{n}{2(2n+1)} & \frac{-1}{2n(2n+1)} & \frac{1}{2n^3} \\ 0 & 0 & \frac{n}{2(2n+1)} & \frac{1}{2n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \zeta(6) + \delta \\ 0 & 0 & 0 & \zeta(4) \\ 0 & 0 & 0 & \zeta(2) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\delta = 9 \sum \frac{H^{(4)}_{k-1}}{\binom{2k}{k} k^2} \approx 0.438668$$

Another suspicious identity

$$\sum_{n \in \mathbb{Z}} n^2 e^{-\frac{n^2}{2}} \stackrel{?}{=} \sum_{n \in \mathbb{Z}} e^{-\frac{n^2}{2}}$$

motivation:

$$\int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$$

Suggesting

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \sum_{n \in \mathbb{Z}} e^{-\frac{n^2}{2}}$$

For example:

$$\int_0^1 x^{-x}\,dx = \sum_{n \geq 1} n^{-n}$$

" Freshman Dream "

About the Freshman Dream identity

- $$\int_0^1 x^{-x}\,dx = \int_0^1\int_0^1 (xy)^{-xy}\,dx\,dy$$

- $$\left|\sum_{n\geq 1} n^{-n} - \frac{1660 + 550\sqrt{\pi} - 15\pi - 16\pi\sqrt{\pi} + 148\pi^2}{976\pi}\right| \leq 10^{-21}$$

- $$\left|\sum_{n\geq 1} n^{-n-1} - \frac{1584742}{691313}\pi + \frac{167}{854}\pi^3\right| \leq 10^{-20}$$

(using PSLQ)

$$\underbrace{\sum_{n\in\mathbb{Z}} n^2 e^{-\frac{n^2}{2}}}_{\sqrt{2\pi}\,-\,5.16\times10^{-7}} \stackrel{?}{=} \underbrace{\sum_{n\in\mathbb{Z}} e^{-\frac{n^2}{2}}}_{\sqrt{2\pi}\,+\,1.32\times10^{-8}}$$

In fact

$$\frac{\sum_{n\in\mathbb{Z}} n^2 e^{-\frac{n^2}{2}}}{\sum_{n\in\mathbb{Z}} e^{-\frac{n^2}{2}}} = \frac{K^2(k)}{\pi}\left[\frac{E(k)}{K(k)} - k'^2\right]$$

$$\simeq 0.9999997887677$$

How to choose c such that

$$\sum_{n\in\mathbb{Z}} n^2 e^{-cn^2} = \sum_{n\in\mathbb{Z}} e^{-cn^2} \ ?$$

$$c = \frac{K'(k)}{K(k)} \quad \text{with} \quad \frac{K^2(k)}{\pi^2}\left[\frac{E(k)}{K(k)} - 1\right] = 1$$

$$c \simeq 0.4999989438$$

Another suspicious identity

$$\sum_{n \in \mathbb{Z}} n^4 e^{-\frac{n^2}{2}} = 3 \sum_{n \in \mathbb{Z}} e^{-\frac{n^2}{2}}$$

motivation:

$$\int_{-\infty}^{\infty} x^4 e^{-\frac{x^2}{2}} \, dx = 3 \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx$$

$$\sum_{n\in\mathbb{Z}} n^4 e^{-\frac{n^2}{2}} = 3\sum_{n\in\mathbb{Z}} e^{-\frac{n^2}{2}}$$

is wrong: in fact

$$\frac{\sum_{n\in\mathbb{Z}} n^4 e^{-\frac{n^2}{2}}}{\sum_{n\in\mathbb{Z}} e^{-\frac{n^2}{2}}} \simeq 3.000000707$$

can be evaluated analytically

$$\frac{\sum_{n\in\mathbb{Z}} n^4 e^{-\frac{n^2}{2}}}{\sum_{n\in\mathbb{Z}} e^{-\frac{n^2}{2}}} = 3\sigma^2 + \frac{1}{8}\,\theta_3^8\left(e^{-\frac{1}{2}}\right)(kk')^2$$

$$\text{with } \sigma^2 = \frac{K^2(k)}{\pi}\left[\frac{E(k)}{K(k)} - k'^2\right]$$

When I was a student, Abelian functions were, as an effect of the Jacobian tradition, considered the uncontested summit of mathematics and each of us was ambitious to make progress in this field.

Felix Klein

A digital high precision fraud

$$n \in \mathbb{N}$$

$$n = n_k n_{k-1} \cdots n_0 \big|_{10}$$

$$= n_k \cdot 10^k + n_{k-1} \cdot 10^{k-1} + \cdots + n_0$$

$$753 = 7 \cdot 10^2 + 5 \cdot 10 + 3$$

define

$$a(n) = \#\{n_k : n_k \text{ even}\}$$

$$b(n) = \#\{n_k : n_k \text{ odd}\}$$

$$n = 3141592$$

$$a(n) = 2$$

$$b(n) = 5$$

define

$$c(n) = 10^5 \times a(n) - \frac{b(n)}{10^5}$$

then

$$\sum_{n \geq 0} \frac{c(n)}{10^n} = \frac{11111111111}{110000}$$

$$\underbrace{\sum_{n \geq 0} \frac{c(n)}{10^n}}_{101010.1010090909\ldots} = \underbrace{\frac{11111111111}{110000}}_{101010.1010090909\ldots}$$

is correct up to 105 decimal places

<u>proof</u> : the generating function for $b(n)$

$$\sum_{n \geq 0} t^{b(n)} x^{n} =$$

$$\prod_{n \geq 0} \left(1 + t \cdot x^{1 \cdot 10^{n}} + x^{2 \cdot 10^{n}} + t \cdot x^{3 \cdot 10^{n}} + \cdots + x^{8 \cdot 10^{n}} + t \cdot x^{9 \cdot 10^{n}}\right)$$

Take the log-derivative in x

$$\frac{\sum_{n \geq 0} b(n)\, t^{b(n)-1} x^{n}}{\sum_{n \geq 0} t^{b(n)} x^{n}}$$

$$= \sum_{n \geq 0} \frac{x^{10^n} + x^{3.10^n} + \cdots + x^{9.10^n}}{1 + t x^{10^n} + x^{2.10^n} + \cdots + t x^{9.10^n}}$$

and evaluate at $t = 1$

$$\sum_{n \geq 0} b(n)\, x^n = \frac{1}{1-x} \sum \frac{x^{10^n}}{1+x^{10^n}}$$

$$= 0 + 1.x^1 + 0.x^2 + 1.x^3$$

$$+ \dots + 1.x^9 + 1.x^{10}$$

$$+ 2.x^{11} + 1.x^{12} + 2.x^{13} + \dots$$

In the same way

$$\sum_{n\geq 0} a(n)x^n = 1 + \frac{1}{1-x}\sum \frac{x^{2.10^n}}{1+x^{10^n}}$$

and, with $c(n) = 10^5 a(n) - \frac{b(n)}{10^5}$, the generating function for c(n):

$$\sum_{n\geq 0} c(n)x^n = 10^5 + \sum_{n\geq 0} \frac{10^5 x^{2\cdot 10^n} - 10^{-5} x^{10^n}}{1+x^{10^n}}$$

Evaluating at $x = \frac{1}{10}$:

the term $(n=1)$ in the sum

$$10^{5}\left(\frac{1}{10}\right)^{20} - 10^{-5}\left(\frac{1}{10}\right)^{10} = 0$$

and all subsequent $n>1$ terms are increasingly small.

$$\sum_{n \geq 0} \frac{c(n)}{10^{n}} = \frac{11111111111}{110000} - \varepsilon \qquad \varepsilon < 10^{-105}$$

Generalization :

$$c_k(n) = k^5 a(n) - \frac{b(n)}{k^5}$$

$k = 100$:

$$\sum_{n \geqslant 0} \frac{c_{100}(n)}{100^n} \simeq \frac{101010101010101010101}{10100000000}$$

$k = 10^p$:

$$\sum_{n \geqslant 0} \frac{c_{10^p}(n)}{10^{pn}} \simeq \frac{[1[0]_{p-1}]_{10}1}{1[0]_{p-1}1[0]_{4p}}$$

$\varepsilon = 10^{-105p}$

Other digital sums

$s_2(n)$ = number of 1's in the binary expansion of n

. $n = 9 = 1.2^3 + 0.2^2 + 0.2^1 + 1.2^0$

$= \underline{1}00\underline{1} \Rightarrow s_2(9) = 2$

- $$\prod_{n \geqslant 1} \left(\frac{1+\frac{1}{n}}{1+\frac{1}{n+1}} \;\; \frac{1+\frac{1}{2n+2}}{1+\frac{1}{2n}} \right) = \frac{3}{4}$$

- $$\prod_{n \geqslant 1} \left(\frac{1+\frac{1}{n}}{1+\frac{1}{n+1}} \;\; \frac{1+\frac{1}{2n+2}}{1+\frac{1}{2n}} \right)^{S_2(n)} = \frac{\pi}{2}$$

$$\prod_{n \geqslant 1} \left(\frac{1+\frac{1}{2n}}{1+\frac{1}{2n+2}} \;\; \frac{1+\frac{1}{4n+4}}{1+\frac{1}{4n}} \right)^{S_2(n)} = \frac{\pi}{2} \prod_{k=1}^{n} \tanh^2\left(k\frac{\pi}{2}\right)$$

$$= 2\sqrt{\frac{2}{\pi}}\, \Gamma^2\left(\frac{5}{4}\right)$$

Discrete Gaussians

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Infinite matrix products

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Borwein litterature

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