A discovery of Ramanujan's Notebooks

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Outline

- Ramanujan's Notebooks
- Ramanujan and telescoping
- integration by differentiation
- Ramanujan's cos cosh identity

A trigonometric Integral

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$$I_{n,p} = \int_0^\infty \frac{\sin^{2n+1} x}{x} \cos(2px)\, dx$$

$$= (-1)^p \frac{\sqrt{\pi}}{2} \frac{\Gamma(n+1)\,\Gamma(n+\frac{1}{2})}{\Gamma(n-p+1)\,\Gamma(n+p+1)}$$

for all n, p integers

A trigonometric Integral

$$I_{n,p} = \int_0^\infty \frac{\sin^{2n+1} x}{x} \cos(2px)\, dx$$

proof :

$$I_{n,p} = \frac{1}{2} I_{n-1,p} - \frac{1}{4} I_{n-1,p+1} - \frac{1}{4} I_{n-1,p-1}$$

$$\begin{tikzcd}
 & I_{n,p} \arrow[dl, "-1/4"'] \arrow[d, "1/2"] \arrow[dr, "-1/4"] & \\
I_{n-1,p-1} & I_{n-1,p} & I_{n-1,p+1}
\end{tikzcd}$$

A trigonometric Integral

Integration by differentiation

Observe:

$$\underbrace{\int_0^\infty e^{-xy}\, y^n\, dy}_{\frac{\Gamma(n+1)}{x^{n+1}}} = \underbrace{(-\partial_x)^n \frac{1}{x}}_{\frac{\Gamma(n+1)}{x^{n+1}}}$$

$$\partial_x = \frac{d}{dx}$$

so that, for f analytic

$$\int_0^\infty e^{-xy} f(y)\, dy = f(-\partial_x)\frac{1}{x}$$

$$\partial_x = \frac{d}{dx}$$

Other versions :

$$\int_a^b e^{-xy} f(y)\,dy = f(-\partial_x) \overbrace{\int_a^b e^{-xy}\,dy}^{\frac{1}{x}\left(e^{-bx} - e^{-ax}\right)}$$

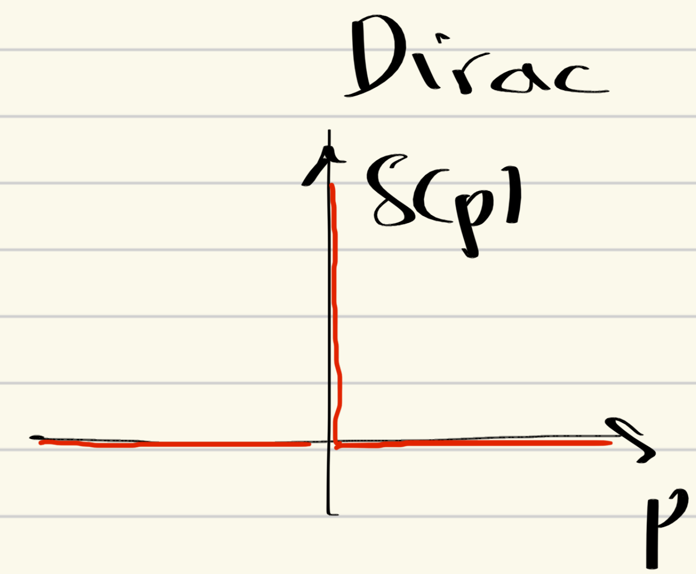
$$= f(-\partial_x)\,\frac{e^{-bx} - e^{-ax}}{x}$$

$$\int_{-\infty}^{\infty} e^{-ixy} f(y)\,dy = f(-\partial_x) \overbrace{\int_{-\infty}^{\infty} e^{-ixy}\,dy}^{2\pi\,\delta(x)}$$

$$= 2\pi\, f(-\partial_x)\,\delta(x)$$

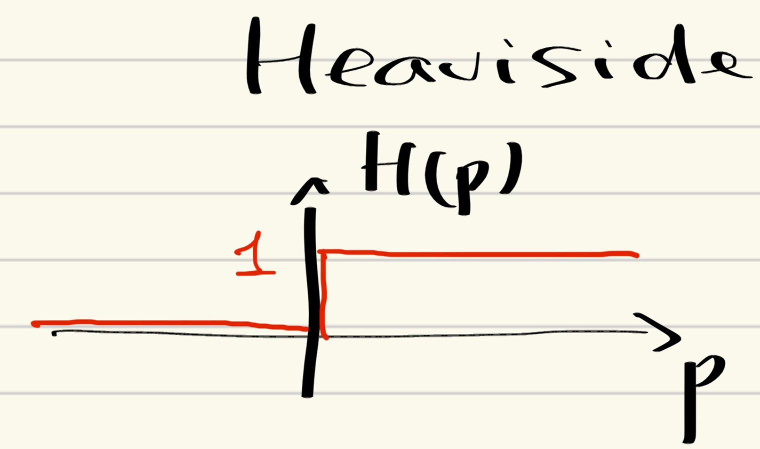
Application :

$$I_{n,p} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^{2n+1}\left(\frac{x}{2}\right)}{x}\, e^{ipx}\, dx$$



$$= \frac{\pi}{i}\, \frac{\sin^{2n+1}\left(i\frac{\partial_p}{2}\right)}{i\partial_p}\, \delta(p)$$

$$= \frac{\pi}{i}\, \sin^{2n+1}\left(i\frac{\partial_p}{2}\right) H(p)$$



and expand $\sin^{2n+1}\left(i\frac{\partial_p}{2}\right) = \left(\frac{e^{i\frac{\partial_p}{2}} - e^{-i\frac{\partial_p}{2}}}{2i}\right)^{2n+1}$

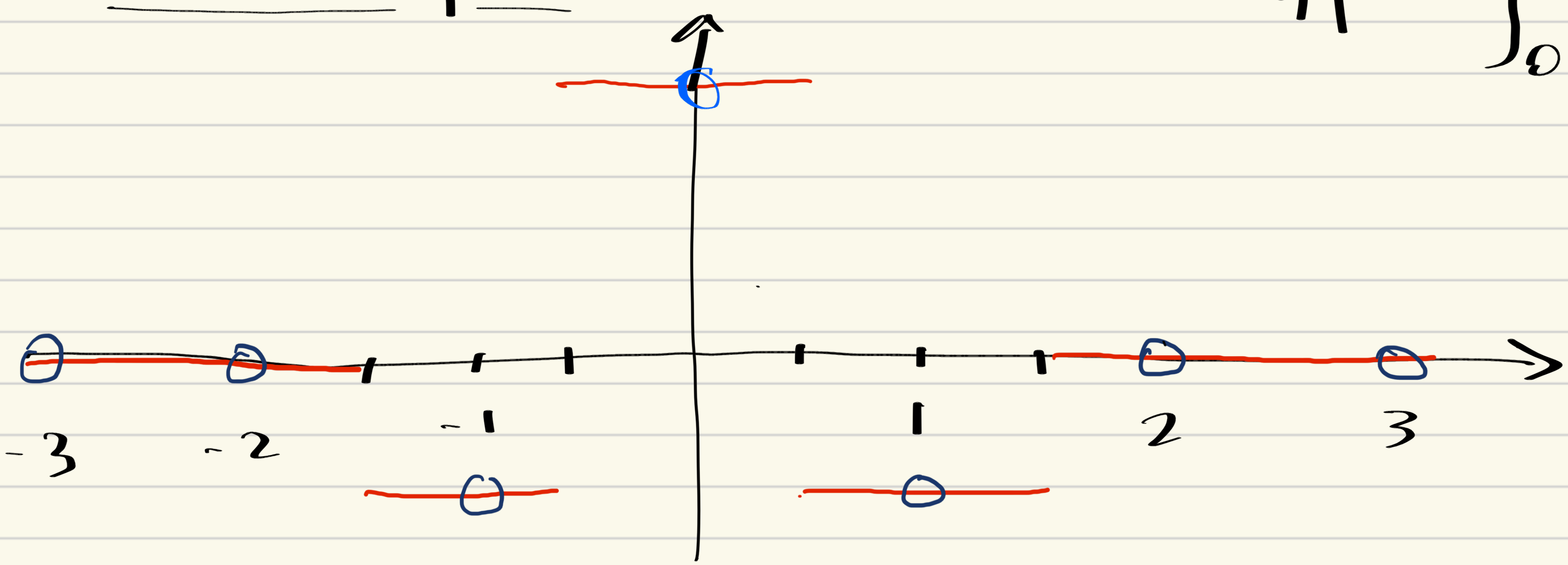
$$I_{np} = \int_0^\infty \frac{\sin^{2n+1} x}{x} \cos(2px)\, dx$$

$$= \frac{(-1)^n \pi}{2^{2n+1}} \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k H\left(p+n+\frac{1}{2}-k\right)$$

a piecewise constant function

<u>Example</u> : $n=1$

$$I_{1,p} = \int_0^\infty \frac{\sin^3 x}{x} \cos(2px)\, dx$$

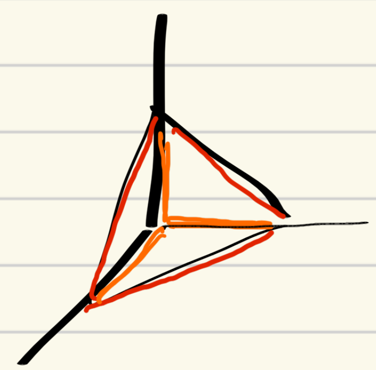


$$= \begin{cases} 0 & |p| > \frac{3}{2} \\ -\frac{\pi}{8} & \frac{1}{2} < |p| < \frac{3}{2} \\ \frac{\pi}{4} & |p| < 1/2 \end{cases}$$

More integration by differentiation

Consider $I_n(\underline{a}) = \int_{S_n} e^{-\underline{x}\underline{a}} \, d\underline{x}$ $\quad \underline{x} \in \mathbb{R}^n$, $\underline{a} \in \mathbb{R}^n$

on the simplex $\quad S_n = \begin{cases} x_1, \dots, x_n \geq 0 \\ x_1 + \dots + x_n \leq 1 \end{cases}$



Write $\quad I_n(\underline{a}) = \int_{\mathbb{R}^n} \mathbb{1}_{S_n}(\underline{x}) \cdot e^{-\underline{x}\underline{a}} \, d\underline{x}$

with the indicator function

$$\mathbb{1}_{S_n}(\underline{x}) = \begin{cases} 1 & \underline{x} \in \underline{S}_n \\ 0 & \text{else} \end{cases}$$

By the method of integration by differentiation

$$I_n(\underline{a}) = \mathbb{1}_{S_n}(\underline{\partial}_a) \int_{[0,\infty)^n} e^{-\underline{x}\,\underline{a}} \, d\underline{x}$$

$$= \mathbb{1}_{S_n}(\underline{\partial}_a) \frac{1}{a_1 a_2 \cdots a_n}$$

Take $n=1$ and use

$$\mathbb{1}_{S_1}(x) = H(1-x) = \begin{cases} 1, & x<1 \\ 0 & \text{else} \end{cases}$$

and $$H(x) = \frac{1}{2}\left(1 + \frac{1}{\pi}\int_0^\infty \frac{\sin(xy)}{y}\,dy\right)$$

$$\sin((1+\partial_a)y)\frac{1}{a} = \frac{1}{2i}\left[\frac{e^{iy}}{a+iy} - \frac{e^{-iy}}{a-iy}\right]$$

$$\int_{\mathbb{R}} \frac{\sin((1+\partial_a)y)}{y}\,dy\;\frac{1}{a} = \int_{\mathbb{R}} \frac{a\sin y - y\cos y}{y(a^2+y^2)}\,dy$$

$$= \pi\,\frac{1-2e^{-a}}{a}$$

and $$\mathcal{I}_{[0,1)}(a) = \int_0^1 e^{-ax}\,dx = \frac{1-e^{-a}}{a}$$

Extends to

$$\int_{S_n} e^{-\underline{x}\,\underline{a}}\, d\underline{x} = \sum_{k=1}^{n} \frac{e^{-a_k} - 1}{(-a_k)\,\varphi'(-a_k)}$$

with $\varphi(x) = \prod_{k=1}^{n} (x + a_k)$

Ramanujan and Telescoping

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$$S = \log 2 \sum_{k\geq 2} \frac{(-1)^k}{k \log k} + \log^2 2 \sum_{k \geq 2} \frac{1}{k \log k \log(2k)} = 1$$

Ramanujan and Telescoping

$$S = \log 2 \sum_{k\geq 2} \frac{(-1)^k}{k \log k} + \log^2 2 \sum_{k \geq 2} \frac{1}{k \log k \log(2k)} = 1$$

proof by B. Berndt : consider

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} + \sum_{k\geq 2} \frac{(-1)^k \log 2}{k \log k} + \sum_{k \geq 2} \frac{\log^2 2}{k \log(2^n k) \log(2^{n+1} k)}$$

and show $S_n = S$ for all n

Moreover,

$$S_\infty = \lim_{n \to \infty} S_n = \sum_{k \geq 1} \frac{1}{k(k+1)} = 1$$

so that $S = 1$.

another proof ?

$$\frac{1}{\log k} = \int_{-\infty}^{0} k^u \, du$$

$$\frac{1}{\log k \log 2k} = \int_{-\infty}^{0} \int_{-\infty}^{0} k^u (2k)^v \, du dv$$

$$= \int_{-\infty}^{0} k^x \int_{-\infty}^{0} 2^y \, dy \, dx$$

$$= \frac{1}{\log 2} \int_{-\infty}^{0} k^x (1 - 2^x) \, dx$$

$$= \frac{1}{\log 2} \left[\frac{1}{\log k} - \frac{1}{\log 2k} \right]$$

we deduce

$$\log 2 \sum_{k\geq 2} \frac{(-1)^k}{k\log k} + \log^2 2 \sum_{k\geq 2} \frac{1}{k\log k \log(2k)}$$

$$= \log 2 \left[\sum_{k\geq 2} \frac{(-1)^k}{k\log k} + \frac{1}{k\log k} - \frac{1}{k\log(2k)} \right]$$

$$\frac{(-1)^k+1}{k\log k} \qquad \frac{2}{2k\log(2k)}$$

$$= \log 2 \left[\sum_{k\geq 2} \frac{1}{k\log(2k)} - \sum_{k\geq 1} \frac{1}{k\log(2k)} \right] = 1$$

Extensions: proof only uses the additivity property

$$\log(xy) = \log x + \log y$$

Let f a completely additive function

$m, n \in \mathbb{Z}$ $\qquad f(m.n) = f(m) + f(n)$

such that $\quad f(n) \neq 0 \quad n \geq 2.$

Then

$$f(2)\sum_{k\geq 1}\frac{(-1)^k}{kf(k)} + f(2)^2\sum_{k\geq 1}\frac{1}{kf(k)f(2k)} = 1$$

Example :

$f(k) = \Omega(k) = \#$ prime divisors of k

Another generalization

- $$\log 2 \sum_{k\geq 2} \frac{(-1)^k}{k\log(2k)} + \log^2 2 \sum_{k\geq 2} \frac{1}{k\log(2k)\log(4k)} = \frac{1}{2}$$

- $$\log 2 \sum_{k\geq 2} \frac{(-1)^k}{k\log(2k)} + \log 2 \sum_{k\geq 2} \frac{(-1)^k}{k\log(4k)}$$
$$+ 2\log^2 2 \sum_{k\geq 2} \frac{1}{k\log(2k)\log(8k)} = \frac{5}{6}$$

More generally, for $q > 1$, $q \in \mathbb{N}$

$$\sum_{\ell=1}^{q} \log 2 \sum_{k \geq 2} \frac{(-1)^k}{k \log(2^\ell k)}$$

$$+ q \log^2(2) \sum_{k \geq 2} \frac{1}{k \log(2k) \log(k \cdot 2^{q+1})}$$

$$= \sum_{\ell=1}^{q} \frac{1}{\ell+1}$$

Another generalization : $z \notin \{-2,-3,\dots\}$

- $$\sum_{k\geq 2} \frac{(-1)^k}{k\log(k+z)} + \sum_{k\geq 1} \frac{\log 2}{k\log(k+z)\left(\log(2k)+z\right)} = \frac{1}{\log 2 + z}$$

- $\alpha = e^z$

$$\sum_{k\geq 2} \frac{(-1)^k}{k\log(\alpha k)} + \sum_{k\geq 1} \frac{\log 2}{k\log(\alpha k)\log(2\alpha k)} = \frac{1}{\log(2\alpha)}$$

Ramanujan and recursivity

vol IV, Entries 24 to 30, p. 396

$$\sum_{k\geq 1} \frac{1}{2^k\left(1+x^{2^{-k}}\right)} = \frac{1}{\log x} + \frac{1}{1-x}$$

proof: start from

$$\frac{1}{1-x} = \frac{1}{(1+\sqrt{x})(1-\sqrt{x})}$$

$$= \frac{1}{2}\frac{1}{1+\sqrt{x}} + \frac{1}{2}\frac{1}{1-\sqrt{x}}$$

Denote $f_-(x) = \dfrac{1}{1-x}$, $f_+(x) = \dfrac{1}{1+x}$

$$f_-(x) = \frac{1}{2} f_+(\sqrt{x}) + \frac{1}{2} f_-(\sqrt{x})$$

$$f_-(\sqrt{x}) = \frac{1}{2} f_+(x^{1/4}) + \frac{1}{2} f_-(x^{1/4})$$

$$\vdots$$

$$f_-(x) = \frac{1}{2} f_+(x^{1/2}) + \frac{1}{4} f_+(x^{1/4}) + \cdots$$

$$+ \lim_{n \to \infty} \frac{1}{2^n} f_-(x^{1/2^n})$$

with $\lim_{n \to \infty} 2^{-n} f_-(x^{2^{-n}}) = \frac{1}{\log(x)}$

Ramanujan and recursivity

A multiplicative version

$$\frac{e^{z}-1}{z} = \prod_{n\geq 1} \frac{1+e^{z/2^n}}{2}$$

Take $z = \frac{d}{dx} = \partial$ and notice that

$$\frac{e^{\partial}-1}{\partial} f(0) = \int_0^1 f(x)\,dx$$

Consider the approximation

$$\frac{e^z - 1}{z} \simeq \prod_{n=1}^{2} \frac{1+e^{z/2^n}}{2}$$

$$\frac{1+e^{0/2}}{2} \cdot \frac{1+e^{0/4}}{2} f(0)$$

$$= \frac{1}{4}\left[f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\right]$$

Riemann sum

Ramanujan and recursivity

Other examples

$$\sum_{k\geq 1} \frac{1}{2^k} \tan\left(\frac{x}{2^k}\right) = \frac{1}{x} - \cot x$$

from

$$\cot x = \frac{1}{2}\cot\left(\frac{x}{2}\right) - \frac{1}{2}\tan\left(\frac{x}{2}\right)$$

Ramanujan and recursivity

Other examples

$$\sum_{k \geq 1} \frac{\sin^3(3^k x)}{3^k} = \frac{3}{4} \sin x$$

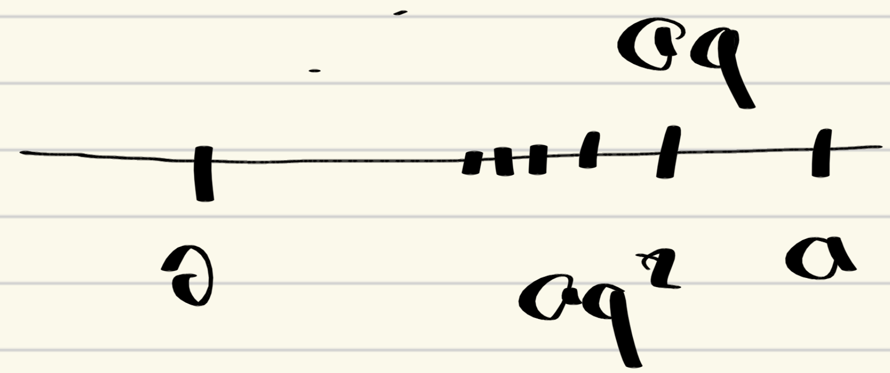
from $$\frac{3}{4} \sin x = \frac{1}{4} \sin(3x) + \sin^3 x$$

Ramanujan and recursivity

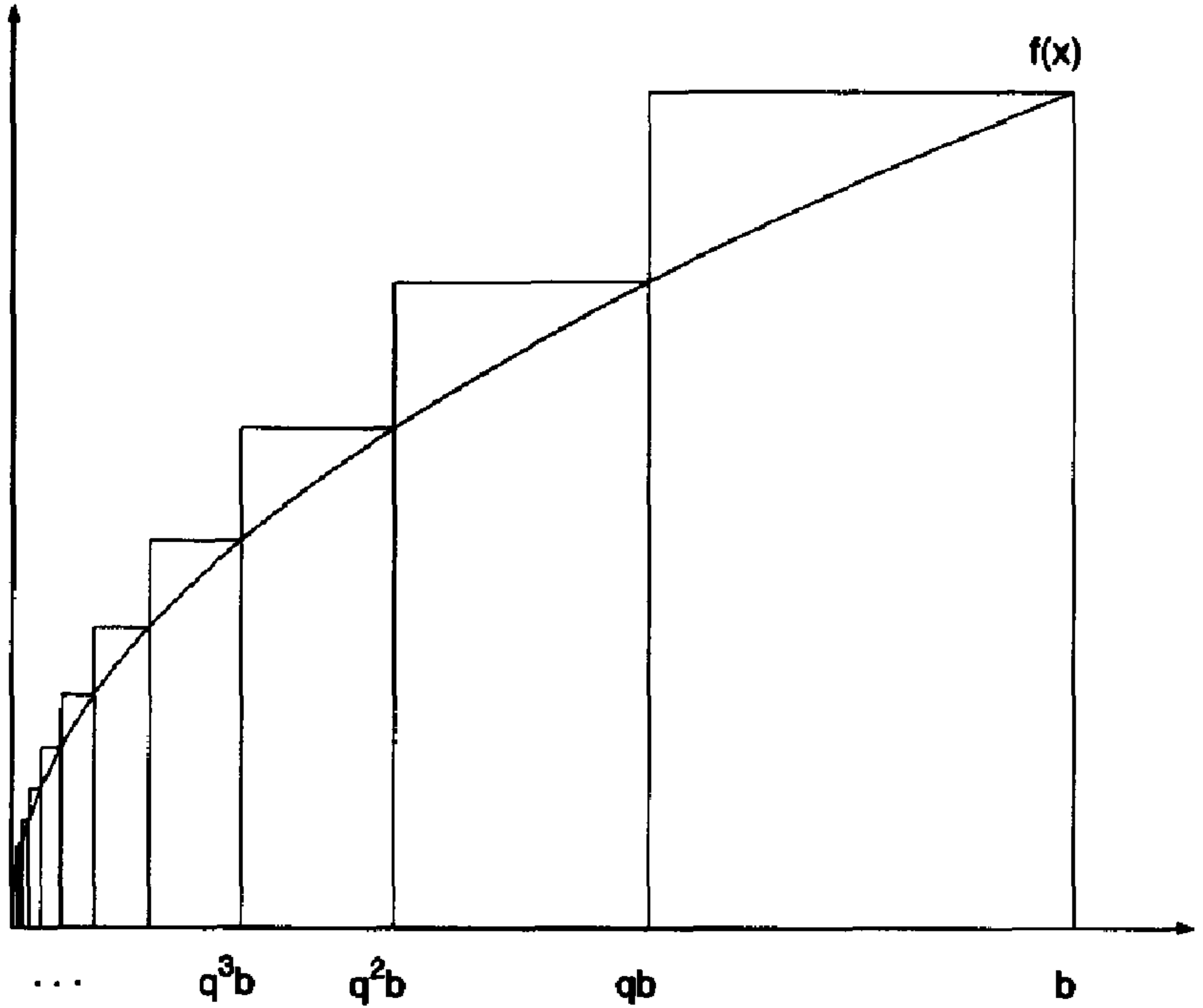
Another application: $|q|<1$

$$\int_0^a f(x)\,d_q x = (1-q)\sum_{n\geq 0} aq^n f(aq^n)$$

Jackson's integral



V. Kac, Quantum Calculus



f(x)
...
q^3b
q^2b
qb
b

For example

$$\sum_{n\geq 1} \frac{1}{2^n\left(1+x^{\frac{1}{2^n}}\right)} = \frac{1}{\log x} + \frac{1}{1-x}$$

is Jackson's integral

$$\int_0^x \frac{1}{1+e^z}\, d_{\frac{1}{2}} z = \frac{1}{2x} + \frac{1}{1-e^{2x}}$$

Ramanujan and trigonometry

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$$(1+x^2)^{\frac{n}{2}} \sin(n \arctan x) = nx \prod_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(1 - \frac{x^2}{\tan^2\left(\frac{k\pi}{n}\right)}\right)$$

proof: based on Chebychev polynomials

Ramanujan and trigonometry

Another proof: assume n even

$$\underbrace{\operatorname{Im}\left(1+ix\right)^{2n}}_{P_{2n}(x)} = (1+x^2)^n \cdot \sin(2n \arctan x)$$

a polynomial in x of degree $n-1$

- $P_{2n}(0) = 0$

- $P_{2n}(x) = Kx \cdot Q_{2n}(x)$ $\qquad K = \lim\limits_{x \to 0} \dfrac{P_{2n}(x)}{x} = 2n$

Ramanujan and trigonometry

Other roots :

$$\mathrm{Im}(1+ix)^{2n} = 0$$

$$(1+ix)^{2n} - (1-ix)^{2n} = 0$$

$$\left(\frac{1+ix}{1-ix}\right)^{2n} = 1 \qquad \frac{1+ix}{1-ix} = e^{i\frac{\pi k}{n}}$$

$$0 \leq k \leq n-1$$

$$x_k = \tan\left(\frac{k\pi}{n}\right) \qquad 0 \leq k \leq n-1$$

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Ramanujan cos cosh identity

$$\left(\sum_{n\in\mathbb{Z}} \frac{\cos(n\theta)}{\cosh(n\pi)}\right)^{-2} + \left(\sum_{n\in\mathbb{Z}} \frac{\cosh(n\theta)}{\cosh(n\pi)}\right)^{-2} = \frac{2}{\pi}\,\Gamma^4\!\left(\frac{3}{2}\right)$$

$$|\theta| < \pi$$

"One wonders how Ramanujan ever discovered this most unusual and beautiful identity"

With Jacobi elliptic function

$$\mathrm{dn}\left(\frac{\theta}{\pi}K,\kappa\right)=\frac{\pi}{2K}\sum_{n\in\mathbb{Z}}\frac{q^n}{1+q^{2n}}\cos(n\theta)$$

This is:

$$\mathrm{dn}^{-2}\left(\frac{\theta}{\pi}K,\kappa\right)+\mathrm{dn}^{-2}\left(\iota\frac{\theta}{\pi}K,\kappa\right)=2$$

$$q=e^{-\pi},\quad \kappa=\frac{1}{\sqrt{2}},\quad K(\kappa)=\frac{1}{2}\frac{\pi^{3/2}}{\Gamma^2(3/4)}$$

Since

$$\mathrm{dn}^{-2}\left(u, \frac{1}{\sqrt{2}}\right) = 1 + \frac{u^2}{2} - \frac{u^6}{40} + \frac{u^{10}}{1200} + \cdots$$

$$\mathrm{dn}^{-2}\left(iu, \frac{1}{\sqrt{2}}\right) = 1 - \frac{u^2}{2} + \frac{u^6}{40} - \frac{u^{10}}{1200} + \cdots$$

$$\mathrm{dn}^{-2}\left(u, \frac{1}{\sqrt{2}}\right) + \mathrm{dn}^{-2}\left(iu, \frac{1}{\sqrt{2}}\right) = 2$$

- $\mathrm{dn}\left(u, \frac{1}{\sqrt{2}}\right)$ is even ✓
- $[u^{4n+2}]\, \mathrm{dn}^{-2}\left(u, \frac{1}{\sqrt{2}}\right) = 0$?

$$[u^{4n+2}]\, dn^{-2}\left(u, \frac{1}{\sqrt{2}}\right) = 0$$

equivalent to

$$\sum_{n \geq 1} (-1)^n \frac{n^{4p+1}}{\sinh(n\pi)} = 0$$

An identity by B. Berndt

$\alpha, \beta > 0 \qquad \alpha\beta = \pi^2$

$$\alpha^{-n} \sum_{k\geq 1} (-1)^{k+1} \frac{\operatorname{csch}(\alpha k)}{k^{2n+1}} = (-\beta)^{-n} \sum_{k\geq 1} (-1)^{k+1} \frac{\operatorname{csch}(\beta k)}{k^{2n+1}}$$
$$+ \; 2^{2n+1} \sum (-1)^k \frac{B_{2k}(\frac{1}{2})}{2k!} \frac{B_{2n+2-2k}(\frac{1}{2})}{2n+2-2k!} \alpha^{n+1-k} \beta^k$$

Take $\alpha = \beta = \pi$, $\quad n = -2p-1 \qquad p \geq 0$

An identity by Ling (1978)

$$\sum_{n\geq 1} \frac{(-1)^{n-1} n^{2s-1}}{\sinh(\pi n c)} = 2\,\frac{(-1)^{s-1}(2s-1)!}{\pi^{2s}}$$

$$\times \sum_{m,n \in \mathbb{Z}^2} \frac{1}{(2m-1+ic(2n-1))^{2s}} + \frac{1}{(2m-1-ic(2n-1))^{2s}}$$

With $c=1$, each double sum vanishes

Extensions of Ramanujan's cos_cosh identity

$$\left(\sum_{n\geq 1} \frac{\sin(2n-1)u}{\sinh(n\pi)}\right)^{-2} - \left(\sum' \frac{\sinh(2n-1)u}{\sinh(n\pi)}\right)^{-2} = \left(\frac{4K}{\pi}\right)^{-2}$$

and

$$\left(\sum_{n\geq 0} \frac{\cos(n+\frac{1}{2})\theta}{\sinh(n+\frac{1}{2})\pi e^{-i\pi/6}}\right)^{-2} + \left(\sum_{n\geq 0}' \frac{\cos(n+\frac{1}{2})\theta e^{i\frac{\pi}{3}}}{\sinh(n+\frac{1}{2})\pi e^{-i\frac{\pi}{6}}}\right)^{-2}$$

$$+\left(\sum_{n\geq 0} \frac{\cos(n+\frac{1}{2})\theta e^{i\frac{2\pi}{3}}}{\sinh(n+\frac{1}{2})\pi e^{-i\pi/6}}\right)^{-2} = \frac{(1+e^{i\frac{\pi}{6}})2^{14/3}}{i\sqrt{3}\,\Gamma^6(\frac{1}{3})}\pi^4$$

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(2) A. Kempf, D.M. Jackson and A. H. Morales, New Dirac delta function based methods with applications to perturbative expansions in quantum field theory, J. Phys. A: Math. Theor. 47 (2014) 415204
(3) C.-B. Ling, Generalization of certain summations due to Ramanujan, SIAM J. Math. Anal. Vol. 9, No. 1, February 1978
(4) V. Kac, P. Cheung, Quantum calculus, Springer, 2002