Cauchy's Integral Formula as an Act of Combinatorics II: Electric Boogaloo

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Ordinary Generating Functions

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- Example: The OGF of $a_n = 1$ is $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

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- If the coefficients of an OGF "count" the number of some "unlabelled" discrete structure, then the algebra of formal power series can directly be interpreted into combinatorial statements about the structure.
- "A generating function is a device somewhat similar to a bag. Instead of carrying many little objects detachedly, which could be embarrassing, we put them all in a bag, and then we have only one object to carry, the bag." – George Polya

Example: The Catalan Numbers

- The Catalan numbers (which count full binary trees, polygon triangulations, etc.) satisfy Segner's recurrence relation

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- If we let $C(x)$ denote the OGF for the Catalan numbers, the above recurrence yields

$$C(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + \sum_{n=0}^{\infty} c_{n+1} x^{n+1} = 1 + x \sum_{n=0}^{\infty} \sum_{j=0}^{n} c_j c_{n-j} x^n$$

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- So $C(x) = 1 + xC(x)^2$. Solving for $C(x)$ and using the fact that $c_0 = 1$ gives us that $\boxed{C(x) = \frac{1-\sqrt{1-4x}}{2x}.}$

Exponential Generating Functions

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- EGFs are used to count "labelled" discrete structures.

- When valuation is desired, the presence of the fast-growing factorials in the denominator helps ensure the convergence of the EGF.

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- The EGF of $b_n = n!$ (a.k.a. the number of permutations of n distinct objects) is

$$h(x) = \sum_{n=0}^{\infty} n! \frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

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- Another example: $[x^k](1+x)^n = \binom{n}{k}$ since

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k. \qquad \text{(by The Binomial Theorem)}$$

Theorem (Cauchy's Coefficient Theorem)

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series. If $f(z)$ is analytic in a region Ω containing 0, then

$$[z^n]f(z) := a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}}\, dz$$

for any simple, positively oriented loop C around 0 in Ω.

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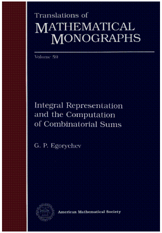
Egorychev's Method

G.P. Egorychev (1938-2023)



- In his book *Integral Representation and the Computation of Combinatorial Sums*, Egorychev outlines a method for simplifying combinatorial sums.

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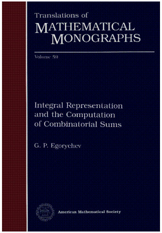

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- In his book *Integral Representation and the Computation of Combinatorial Sums*, Egorychev outlines a method for simplifying combinatorial sums.
- The idea is to identify terms that can be summed in closed form by replacing certain factors with contour integrals.

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Answer: Since $\binom{n}{k} = [z^k](1+z)^n$,

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k}^2 &= \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{1}{2\pi i} \int_C \frac{(1+z)^n}{z^{k+1}}\, dz \\ &= \frac{1}{2\pi i} \int_C (1+z)^n \left(1 + \frac{1}{z}\right)^n \frac{1}{z}\, dz \\ &= \frac{1}{2\pi i} \int_C \frac{(1+z)^{2n}}{z^{n+1}}\, dz \\ &= [z^n](1+z)^{2n} = \boxed{\binom{2n}{n}}\end{aligned}$$

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$$\frac{1}{2\pi i} \int_C (1+z)^n \left(1 + \frac{1}{z}\right)^n \frac{1}{z}\, dz,$$

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- This suggests a "double-counting" explanation.

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1. **RHS:** You can adopt n pets out of $2n$ total in $\binom{2n}{n}$ ways.
2. **LHS:** Condition on the number of kittens you want. If you adopt k kittens with $0 \leq k \leq n$, then you will have $n - k$ puppies. Summing across all possible k yields

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} = \sum_{k=0} \binom{n}{k}^2.$$

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1. Apply Egorychev's method to rewrite a combinatorial expression as a contour integral.
2. Simplify the integral to attain the coefficient of a well-known generating function.
3. Translate the symbolic algebra of generating functions back to counting methods.

Common Integral Representations

Sequences lower down this list have precedence:

- Choosing k objects from n total, without replacement:

$$\binom{n}{k} = \frac{1}{2\pi i}\int_C \frac{(1+z)^n}{z^{k+1}}\,dz = \frac{1}{2\pi i}\int_C \frac{1}{(1-z)^{k+1}z^{n-k+1}}\,dz$$

- Choosing k objects from n total, with replacement:

$$n^k = \frac{k!}{2\pi i}\int_C \frac{e^{nz}}{z^{k+1}}\,dz$$

- Iverson bracket (indicator function for whether $k \le n$):

$$[[k \le n]] = \frac{1}{2\pi i}\int_C \frac{z^k}{z^{n+1}}\frac{1}{1-z}\,dz$$

Common Integral Representations (cont'd)

Sequences lower down this list have precedence:

- Stirling numbers of the second kind (number of set partitions of $\{1, \ldots, n\}$ with k blocks):

$$\left\{ {n \atop k} \right\} = \frac{n!}{k!} \cdot \frac{1}{2\pi i} \int_C \frac{1}{z^{n+1}} (e^z - 1)^k \, dz$$

- Stirling numbers of the first kind (number of permutations in S_n with k cycles):

$$\left[{n \atop k} \right] = \frac{n!}{k!} \cdot \frac{1}{2\pi i} \int_C \frac{1}{z^{n+1}} \left(\log \frac{1}{1-z} \right)^k dz$$

Example 2: $\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^n = ?$

2) Evaluate $\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^n$. Interpret combinatorially.

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2) Evaluate $\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^n$. Interpret combinatorially.

Answer: According to the rule of thumb from earlier,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\cdot\frac{n!}{2\pi i}\int_C \frac{e^{(n-k)z}}{z^{n+1}}\,dz$$

$$= \frac{n!}{2\pi i}\int_C \frac{(e^z-1)^n}{z^{n+1}}\,dz$$

$$= n![z^n](e^z-1)^n = \boxed{n!}$$

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- $\binom{n}{k}$ hints towards choosing k of the n, and the $(n-k)^n$ hints towards arranging (with repetition) the *remaining* numbers
- the generating function $(e^z - 1)^n$ suggests we need to take "nonempty" selections of each of the numbers in $[n]$

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- The RHS of $n!$ is obvious.
- For the LHS, take the complement of the set of functions on that are *not* surjective.
- If $A_k = \{f : [n] \to [n] \mid \mathrm{Im} f \text{ is missing } k \text{ numbers}\}$, then by the principle of inclusion-exclusion

$$n^n - \sum_{k=1}^{n}(-1)^{k-1}|A_k| = n^n - \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(n-k)^n$$
$$= \sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^n$$

Example 3: $\sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{n+k}\begin{bmatrix} n+k \\ k \end{bmatrix} = ?$

3) Evaluate $\sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{n+k}\begin{bmatrix} n+k \\ k \end{bmatrix}$. Interpret combinatorially.

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Answer Sketch: Using the same rule of thumb as before,

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{n+k}\left[{n+k \atop k}\right] \\
=&\sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{n+k}\frac{(n+k)!}{k!}\cdot\frac{1}{2\pi i}\int_C \frac{[-\log(1-z)]^k}{z^{n+k+1}}\,dz \\
&= \frac{(2n)!}{n!}\cdot\frac{1}{2\pi i}\int_C \frac{(-\log(1-z)-z)^n}{z^{2n+1}}\,dz \\
&= \frac{(2n)!}{n!}[z^{2n}](-\log(1-z)-z)^n = \frac{(2n)!}{n!\cdot 2^n} = \boxed{(2n-1)!!}
\end{aligned}$$

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- Such a permutation must be comprised entirely of cycles of length 2. There are $(2n-1)!!$ such permutations.

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- Such a permutation must be comprised entirely of cycles of length 2. There are $(2n-1)!!$ such permutations.
- It can be shown that the LHS is using inclusion-exclusion to count the complement: permutations of $[2n]$ with n cycles that have fixed points.

Exercises: Evaluate and interpret combinatorially

1. $\sum_{k=0}^{r} \binom{n}{k} \binom{m}{r-k}$ **(easy)**

2. $\sum_{k=1}^{n} \binom{n}{k} \left\{ n \atop k \right\} k!$ **(medium)**

3. $\sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \binom{n}{k} \binom{m-2k+n-1}{n-1}$ where $m \leq n$ **(hard)**

4. $\sum_{k=0}^{n} 2^{-k} \binom{n+k}{k}$ [*Hint:* Iverson bracket] **(for the brave)**