Local-Global Principle of Elliptic Curves

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Local-Global Principle

"Description": Let V be a system of Diophantine equations with coefficients over a number field K. The system V has a solution over K if and only if it has a solution modulo $\mathfrak{p}$ for all primes (finite and infinite).

Example

Question: Find the integer solutions of the equation $y^2 = 3x^3 + 2$?

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Example

Question: Find the integer solutions of the equation $y^2 = 3x^3 + 2$?

Solution: It is enough to consider the equation modulo 3.

Local-Global Principle

Example (Legendre)

The quadratic form $ax^2 + by^2 + cz^2 = 0$ has an integer solution if and only if the system

$$\begin{aligned} u^2 &\equiv -bc \pmod{|a|} \\ v^2 &\equiv -ac \pmod{|b|} \\ w^2 &\equiv -ab \pmod{|c|}. \end{aligned}$$

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Example (Selmer)

The equation

$$3x^3 + 4y^3 + 5z^3 = 0$$

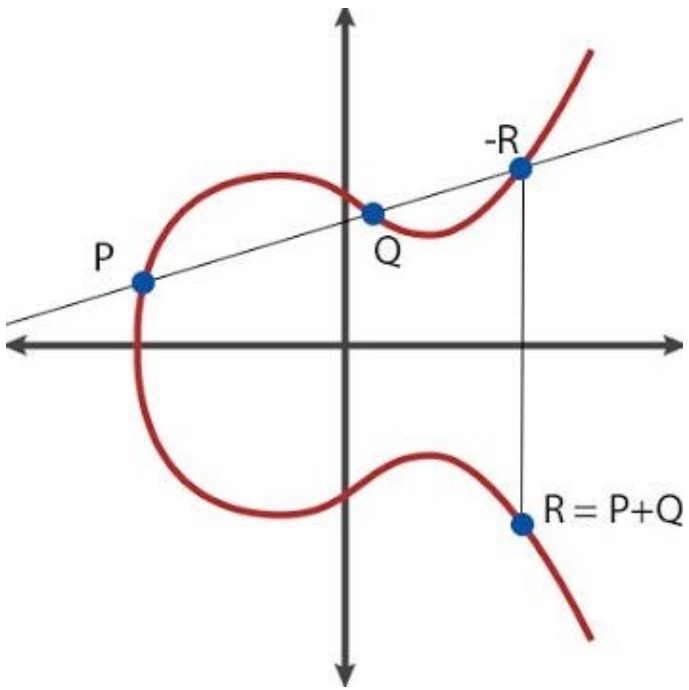
has a solution over $\mathbb{R}$ and modulo p for all primes but it does **not** have an integer solution.

Elliptic Curves

Let K be a field. Then an elliptic curve E/K is a non-singular curve of the form

$$E: \ y^2 = x^3 + Ax + B, \quad A, B \in K.$$

We can give a group structure on E.



Elliptic Curves Isogeny

Let $E,\ E'$ two elliptic curves over K. A homomorphisms $\phi:\ E \to E'$ with $\ker(\phi) < \infty$ is called an **isogeny**.

Definition

We say that E **admits a K-rational ℓ-isogeny** if there exists ϕ as above such that $\#\ker(\phi) = \ell$ and $\ker(\phi)$ is stable under the action of G_K.

Suppose K is a number field and E/K. It is easy to show that when E admits a K-rational ℓ-isogeny then $\tilde{E}/\mathbb{F}_{\mathfrak{p}}$ also admits an $\mathbb{F}_{\mathfrak{p}}$-rational ℓ-isogeny for almost all primes $\mathfrak{p}$ in K, where $\tilde{E}/\mathbb{F}_{\mathfrak{p}}$ is the reduction curve of E at $\mathfrak{p}$.

Local-Global Isogeny Property

Question

When $\tilde{E}/\mathbb{F}_{\mathfrak{p}}$ admits an $\mathbb{F}_{\mathfrak{p}}$-rational ℓ-isogeny for almost all primes $\mathfrak{p}$ in K, does E admit a K-rational ℓ-isogeny?

Local-Global Isogeny Property

In 2012, Sutherland shows that the answer is usually "yes", but there are pairs $(E/K, \ell)$ for which that answer is "no". In particular, the answer depends only on the j-invariant of E and the prime ℓ.

Definition

A pair (j_0, ℓ) with $j_0 \in K$ is called *exceptional for* K is there exists E/K with $j(E) = j_0$ and the answer to the above question is "no". Such a prime ℓ is called an *exceptional prime for* K.

Let ℓ an integer prime with $(\ell, \mathrm{char}(K)) = 1$. We denote by $E[\ell]$ the points of E with order ℓ. We can show that

$$E[\ell] \simeq (\mathbb{Z}/\ell\mathbb{Z})^2 .$$

For $\sigma \in G_K$, the absolute Galois group of K, and $P \in E[\ell]$ it holds that $P^\sigma \in E[\ell]$. Therefore, we get a representation

$$\bar{\rho}_{E,\ell} : \ G_K \to \mathrm{GL}_2(\mathbb{F}_\ell),$$

which is the action of G_K on $E[\ell]$.

Conditions on Exceptional Primes

We can give necessary conditions on exceptional primes. Let $\ell^* = \left(\frac{-1}{\ell}\right)\ell$. Let $G_{E,\ell} = \bar{\rho}_{E,\ell}(G_K)$ and $H_{E,\ell} = \mathbb{P}(G_{E,\ell})$.

Theorem (Sutherland, 2012)

Let K with $\sqrt{\ell^*} \notin K$. If (j_0, ℓ) is exceptional pair for K, then for the elliptic curve E/K with $j(E) = j_0$ holds:

1. *Then $H_{E,\ell} \simeq D_{2n}$, where $n > 1$ is an odd divisor of $(\ell - 1)/2$,*
2. *$\ell \equiv 3 \pmod 4$,*
3. *The group $G_{E,\ell}$ is contained in the normaliser of a split Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_\ell)$,*
4. *E obtains a rational ℓ-isogeny over $K(\sqrt{\ell^*})$.*

Conditions on Exceptional Primes

Theorem (Banwait-Cremona, 2014)

Let K with $\sqrt{\ell^*} \in K$. If (j_0, ℓ) is exceptional pair for K, then for the elliptic curve E/K with $j(E) = j_0$ holds:

1. $H_{E,\ell} \simeq A_4$ *and* $\ell \equiv 1 \pmod{12}$,
2. $H_{E,\ell} \simeq S_4$ *and* $\ell \equiv 1 \pmod{24}$,
3. $H_{E,\ell} \simeq A_5$ *and* $\ell \equiv 1 \pmod{60}$,
4. $H_{E,\ell} \simeq D_{2n}$ *and* $\ell \equiv 1 \pmod{4}$, *where* $n > 1$ *is a divisor of* $(\ell - 1)/2$, *and* $G_{E,\ell}$ *lies in a normaliser of a split Cartan subgroup of* $\mathrm{GL}_2(\mathbb{F}_\ell)$.

Exceptional Primes over $\mathbb{Q}$

Using the above theorem Sutherland proves that the only exceptional pair when $K = \mathbb{Q}$ is $(\frac{2268945}{128}, 7)$. In particular, the elliptic curve

$$y^2 = x^3 - 138915x - 18932130,$$

admits a 7-isogeny at every prime p of good reduction (and over $\mathbb{R}$) but it does not admit a 7-isogeny over $\mathbb{Q}$.

Exceptional Primes over Number Fields

We can search for exceptional primes in the following directions:

1. Either we fix K and try to find all exceptional primes ℓ for K, or
2. we fix ℓ and search if ℓ is exceptional in a "suitable" (infinite) family of number fields K.

Exceptional Primes over Number Fields

Theorem (Anni, 2014)

Let K a number field of degree d and discriminant Δ. Let $\ell_K := \max\{|\Delta|, 6d+1\}$. Then, if (j_0, ℓ) is an exceptional pair for K then it holds

1. $\ell \leq \ell_K$,
2. *There are finitely many exceptional pairs (j_0, ℓ) with $7 < \ell \leq \ell_K$.*

Conjecture (Banwait-Cremona, 2014)

11 is not an exceptional prime for any quadratic field.

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Answer: It is enough to determine the quadratic points of the modular curves $X_{D_{10}}$.

Modular Curves

Let $G \subset \mathrm{GL}_2(\mathbb{F}_\ell)$ with $\det(G) = \mathbb{F}_\ell^*$ and

$$\pi : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{F}_\ell), \tag{1}$$

the natural projection. We denote by $\Gamma := \pi^{-1}(G \cap \mathrm{SL}_2(\mathbb{F}_\ell))$. We define the compact Riemann surface

$$X_G := (\mathbb{H} \cup \{i\infty\} \cup \mathbb{Q})/\Gamma,$$

where $\mathbb{H}$ is the Poincare upper half plane where the action of Γ is given by Möbius transformations. Then, X_G is a curve defined over $\mathbb{Q}$.

Moduli Interpretation

Becaus $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ the natural map $j : X_G \to X(1) \simeq \mathbb{P}^1$ is called the *j-map*. The curve X_G has the following moduli interpretation.

Moduli Interpretation

Every point $P \in X_G(K)$ corresponds to an elliptic curve E/K such that $j(P) = j(E)$ and $G_{E,\ell} \subset G$ and vice versa.

Why Quadratic Points on $X_{D_{10}}$?

Let $\ell = 11$ and K a quadratic field. By Sutherland's theorem we know that the exceptional pairs $(j_0, 11)$ correspond to curves E/K such that $H_{E,11} \simeq D_{10} \subset \mathrm{PGL}_2(\mathbb{F}_{11})$. Let G be the pullback of D_{10} in $\mathrm{GL}_2(\mathbb{F}_{11})$ and $X_{D_{10}} := X_G$.

Hence it is enough to compute the quadratic points of $X_{D_{10}}$.

Question

How do we compute the quadratic points on $X_{D_{10}}$?

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Answer: With the (symmetric) Chabauty method!

p-adic Numbers

Let p a prime. Then any $x \in \mathbb{Q}$ is written in the form $x = p^n \frac{a}{b}$ with $(ab, p) = 1$ and $n \in \mathbb{Z}$. We define the p-adic metric

$$|x| = p^{-n}.$$

The completion of $\mathbb{Q}$ with respect to the p-adic metric is called *the field of p-adic numbers* $\mathbb{Q}_p$. An element $x \in \mathbb{Q}_p$ is of the form

$$x = \sum_{i=-n_0}^{\infty} a_i p^i, \quad a_i \in \{0, \cdots, p-1\}, \ n_0 \in \mathbb{Z}.$$

We can do *p-adic analysis* in $\mathbb{Q}_p$.

p-adic Integration

Suppose X a smooth curve over $\mathbb{Q}$ of genus g and J its Jacobian. Let p be a prime of good reduction for X and $\tilde{X}$ the reduction of X modulo p. We also assume there exists $P_0 \in X(\mathbb{Q})$.

If $\Omega_{J_{\mathbb{Q}_p}}$ is the $\mathbb{Q}_p$-space of global 1-forms of J which has *dimension* g. Then due to Coleman there exists a pairing

$$\Omega_{J_{\mathbb{Q}_p}} \times J(\mathbb{Q}_p) \to \mathbb{Q}_p, \quad (\omega, D) \mapsto \int_0^D \omega,$$

which is bilinear and nondegenarate.

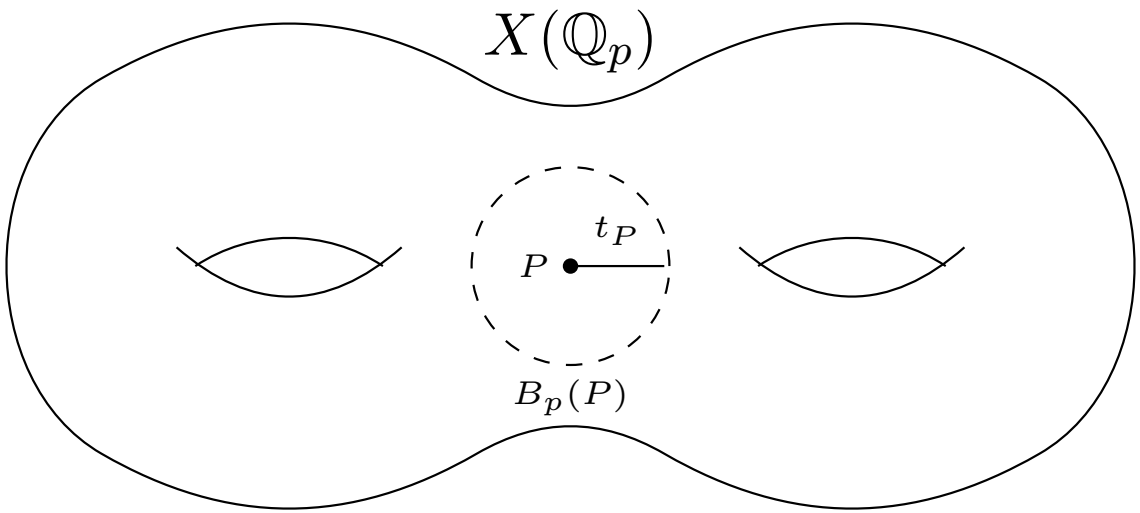
Residue Class

Let $P \in X(\mathbb{Q}_p)$ and $\tilde{P} \in \tilde{X}(\mathbb{F}_p)$ the reduction of P. The *residue class of* P is

$$B_p(P) := \{Q \in X(\mathbb{Q}_p) : \ \tilde{Q} = \tilde{P}\}.$$

Let t_P is a rational function on X that reduces to a uniformizer on $\tilde{X}$ at $\tilde{P}$. It holds

- t_P is a uniformizer at P and $\tilde{t}_{\tilde{P}}$ is a uniformizer at $\tilde{P}$.
- t_P defines a bijection $B_p(P) \to p\mathbb{Z}_p$ and $Q \mapsto t_P(Q)$. Moreover, $t_P(Q) = 0$ if and only if $P = Q$.



Integration on a Residue Class

Proposition

Let X, p, P, t_P as above and $\omega \in \Omega_{J_{\mathbb{Q}_p}}$. There exists a power series

$$\phi(x) = a_1 x + a_2 x^2 + a_3 x^3 + \cdots \in \mathbb{Q}_p[[x]],$$

such that

$$\int_0^{[Q-P]} \omega = \phi(z),$$

for all $Q \in B_p(P)$ and $z = t_P(Q)$.

Chabauty's Idea

Let $\text{rank}(J(\mathbb{Q})) = r$ with $r < g$ (*Chabauty condition*). Suppose $S \subset X(\mathbb{Q})$ a set of known points.

Goal

Prove that $S = X(\mathbb{Q})$.

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Goal

Prove that $S = X(\mathbb{Q})$.

Let $P \in S$ and $Q \in X(\mathbb{Q}) \cap B_p(P)$. Because $r < g$ there exists non-zero $\omega \in \Omega_{J_{\mathbb{Q}_p}}$ such that annihilates $J(\mathbb{Q})$. In other words, there exists ω such that

$$\phi(z) = \int_0^{[Q-P]} \omega = 0.$$

Hence, the only we have to do is to determine the zeros of $\phi(z)$.

Chabauty's Idea in Practice

In practice we prove that $X(\mathbb{Q}) \cap B_p(P) = \{P\}$ for every $P \in S$ and every $p \in T$ where T is a suitable finite set of primes.

Therefore it is enough to show that $X(\mathbb{Q}) \cap B_p(P) = \emptyset$ for all $P \in X(\mathbb{Q}_p) \setminus S$ and $p \in T$.

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But how?

Mordell-Weil Sieve

We can do it using the *Mordell-Weil Sieve*. The idea of the sieve is based on the following commutative diagram.

$$\begin{array}{ccc} X(\mathbb{Q}) & \xrightarrow{\pi} & J(\mathbb{Q}) \\ \downarrow{\scriptstyle red} & & \downarrow{\scriptstyle red} \\ \tilde{X}(\mathbb{F}_p) & \xrightarrow{\pi} & \tilde{J}(\mathbb{F}_p) \end{array}$$

We get $\pi(X(\mathbb{Q})) \subset W_p + L_p$ where W_p is a set of coset representatives of $\mathrm{red}^{-1}(\pi(\tilde{X}(\mathbb{F}_p)))$ and $L_p := \ker\Big(J(\mathbb{Q}) \to \tilde{J}(\mathbb{F}_p)\Big)$.

Let

$$W_T + L_T = \bigcap_{p \in T} (W_p + L_p),$$

where $L_T = \bigcap_{p \in T} L_p$ and W_T a finite subset of $J(\mathbb{Q})$. Obviously, $\pi(X(\mathbb{Q})) \in W_T + L_T$. If T is chosen carefully such that

$$\pi(S) + L_T = W_T + L_T \supset \pi(X(\mathbb{Q})),$$

$$L_T \subset \ker \left(J(\mathbb{Q}) \to \prod_{p \in T} \tilde{J}(\mathbb{F}_p) \right),$$

then we get $S = X(\mathbb{Q})$.

Symmetric Chabauty

Let $P \in X(K)$ where K is a quadratic field and $\bar{P} = P^\sigma$ where σ is a generator of $\mathrm{Gal}(K/\mathbb{Q})$.

Let $X^{(2)}$ the 2-nd symmetric power of X. The elements of $X^{(2)}(\mathbb{Q})$ corresponds to effective $\mathbb{Q}$-rational divisors on X of degree 2. In particular, $\{P, \bar{P}\}$ corresponds to an element of $X^{(2)}(\mathbb{Q})$ and vice versa.

Using the map $X^{(2)} \to J$, $\{P, \bar{P}\} \mapsto [P + \bar{P} - 2P_0]$ we can apply the Chabauty method at $X^{(2)}$ under the assumption that $r < g - 2$ due to the work of Siksek.

The Modular Curve $X_{D_{10}}$

The modular curve $X_{D_{10}}$ has genus 6. By the work of Galbraith, Box and Assaf we can compute a non-singular model which is given by

$$
\begin{aligned}
&uw - 2vw + 2ux - 6vx + 2uy + 2vy + uz = 0,\\
&uw + vw + 2ux - 2vx + 2uy - 10vy - 5uz + 11vz = 0,\\
&-6u^2 + 6uv - 3v^2 + 11w^2 - 66wx + 11x^2 + 88wy - 110xy + 99y^2 + 44wz - 110xz = 0,\\
&6u^2 + 12uv + 12v^2 + 187wx + 22x^2 + 55wy - 44xy - 154y^2 + 66wz + 77xz + 121yz = 0,\\
&-9v^2 + 88w^2 - 11wx - 99x^2 - 77wy + 110xy - 11y^2 + 77wz - 297xz + 121yz = 0,\\
&-6u^2 - 12uv - 12v^2 + 33w^2 - 77wx + 66x^2 - 121wy - 132xy - 110y^2\\
&-44wz - 187xz + 121yz + 121z^2 = 0
\end{aligned}
$$

It holds that $\operatorname{rank}(J_{X_{D_{10}}}(\mathbb{Q})) = 1$.

The Modular Curve $X_{D_{10}}$

The curve $X_{D_{10}}$ is a degree 2 cover of the curve

$$C: \ y^2 + (x^3 + x^2 + x + 1)y = -2x^5 + 2x^4 - 3x^3 + 2x^2 - 2x,$$

under a map $\phi_{X_{D_{10}}} : X_{D_{10}} \to C$ that we can explicitly compute.

Note: It also holds $\text{rank}(J_C(\mathbb{Q})) = 1$. Moreover, $C \simeq X_0^+(121)$.

Proposition

We can prove that $C(\mathbb{Q}) = \{(1, -3), (1, -1), (0, -1), (0, 0), \pm\infty\}$.

Quadratic Points on $X_{D_{10}}$

In particular, the set

$$\begin{aligned} S = \{&(-3/4, 1/4, 0, \pm\frac{\sqrt{77}}{2}, 0, 1),\ (3/4, -5/4, 0, \pm\frac{\sqrt{77}}{2}, 0, 1) \\ &(1, 1, 1, \pm\sqrt{-11}, \pm\sqrt{-11}, 1),\ (-2/5, 2/5, 1/5, \pm\frac{\sqrt{209}}{5}, \mp\frac{\sqrt{209}}{5}, 1), \\ &(-1, 7, 5, \pm\sqrt{473}, \mp\sqrt{473}, 1), (-1/3, 0, -1/3, \pm\frac{\sqrt{22}}{3}, \pm\frac{\sqrt{22}}{3}, 1)\}, \end{aligned}$$

are quadratic points on $X_{D_{10}}$, all are pullbacks of $C(\mathbb{Q})$ under $\phi_{X_{D_{10}}}$.

Chabauty Method on $X^{(2)}_{D_{10}}$

We apply symmetric Chabauty for $p = 5, 7, 13, 17$ and we prove that the elements in S are the only elements in there residue classes modulo p of $X^{(2)}_{D_{10}}(\mathbb{Q}_p)$.

With the Mordell-Weil sieve we prove that S corresponds to the complete set of points on $X^{(2)}_{D_{10}}(\mathbb{Q})$.

The image of the points in S under the j-map is

$$J = [\infty, -3375, 8000, -884736, 16581375, -884736000].$$

The above values do not correspond to exceptional pairs over any quadratic field $K/\mathbb{Q}$.

Main Theorem

From the above we have given a positive answer in the conjecture of Banwait-Cremona.

Theorem (Gajović-Hanselman-K.)

11 *is not an exceptional prime for any quadratic field.*

Future Work

- Prove that 11 is not an exceptional prime for all cubic and quintic number fields.
- Consider exceptional pairs over cubic fields. The only possible primes primes are $\ell = 11$ and 19. We have to determine the cubic points on $X_{D_{10}}$ and $X_0^+(19^2)$. It holds that $g(X_0^+(19^2)) = 9$ and the rank of the Jacobian is 8 (big!).
- Compute quadratic and cubic points on the modular curves in the LMFDB database.

Thank you for your attention!!!