From the first soliton until today

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Do all waves "break"?

Figure: Left: Mark Mathews winning the Australian Big Wave award. Botany Bay, Sydney, Australia. Right: Oregon coast.

Solitons and their history

What is a soliton?

Solitons are special solitary waves which interact elastically, meaning that after the interaction between two or more solitons the individual waves recover their original shape, amplitude, and velocity, and the only effect of the interaction is a position shift.

The first soliton

Solitary waves were first observed by the Scottish engineer John Scott Russell (1808-1882) in 1834, while riding on horseback beside the narrow Union Canal near Edinburgh, Scotland.

While conducting experiments to determine the most efficient design for canal boats, John Scott Russell made a remarkable discovery. It is described in his Report on waves (1844).

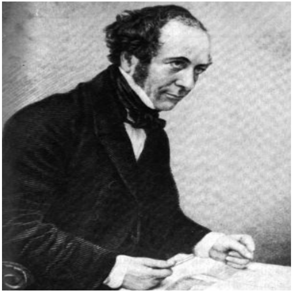


Figure: John Scott Russell

Report on waves (1844)

I was observing the motion of a boat drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel. It rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon...

Request to mathematicians

"...it was not to be supposed that after its existence had been discovered and phenomena determined, endeavors would not be made to reconcile it with existing theory, or to show how it ought to have been predicted from the known equations of fluid motion. In other words, it now remained for the mathematicians to predict the discovery after it had happened..."

Russell's experiments

After his observation, Russell built wave tanks at his home and carried out experiments to study this phenomenon more carefully.

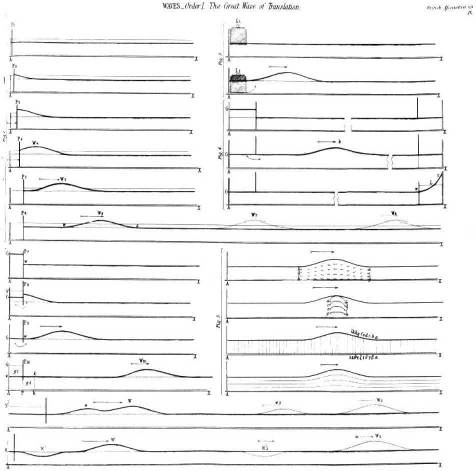


Figure: Russell's experiments

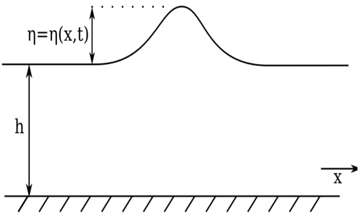
We include some of his results:

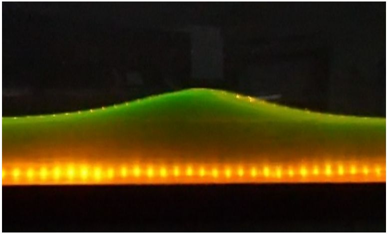
1. He observed stable shallow-water waves that can travel long distances, and thus he deduced that solitary waves exist.
2. Their speed of propagation depends on the size of the wave, i.e., taller waves travel faster, and their width depends on the depth of the water.
3. Unlike regular waves they will never merge. A small solitary wave is overtaken by a large one, after their interaction they split and they recover their original shape, amplitude and velocity. The only effect of the interaction is a position shift. We call this interaction trivial or elastic.

Russell's experiments: recreation in Edinburgh

Figure: Recreation of a solitary wave on the Union Canal.

1. Airy (1845) and Stokes (1847) did not believe Russell's observation. They believed that all waves can be explained by the linear water wave theory.
2. In 1895, two Dutch mathematicians, Korteweg and de Vries published a milestone work in the history of the development of soliton theory. They derived a model for the evolution of long unidirectional waves in shallow water, a nonlinear PDE that approximately described the wave elevation $\eta(x, t)$ above mean height (h).





The first soliton equation!

KdV equation and its solitary wave solution

$$u_t + 6\mathrm{sign}(\gamma)uu_x + u_{xxx} = 0, \quad \gamma = 1 - \frac{3T}{gh^2} \tag{1}$$

where g is the gravity, T is the surface tension, and h is the mean height. The KdV soliton is given by

$$u_s(x,t) = \mathrm{sign}(\gamma)2k^2 sech^2 k\left(x - x_0 - 4k^2 t\right) \tag{2}$$

with

$$|u_{\max}| = 2k^2, \quad \text{speed} = 2|u_{\max}|. \tag{3}$$

1. In 1965, Zabusky and Kruskal discovered that solitary wave solutions of KdV interact elastically.
2. Due to this behavior, Zabusky and Kruskal termed these solutions solitons.
3. Speed and amplitude are preserved upon interaction. The only effect is a position shift.

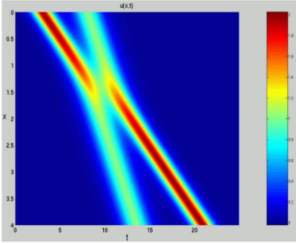


Figure: Soliton interactions

Integrability and IST

Wikipedia: In mathematics, the IST is a method for solving some nonlinear PDEs. ***It is one of the most important developments in mathematical physics in the past 40 years.***

There exists a subclass of nonlinear PDEs that possess deep mathematical structure and require a separate investigation. They are now known as integrable systems, and they admit some common properties:

- They admit traveling wave solutions, often in the form of solitons.
- They can be linearized using the inverse scattering transform (IST).
- They possess an infinite number of conserved quantities.

The classical IST: NLS equation

- The nonlinear Schrödinger (NLS) equation

$$iq_t + q_{xx} - 2\nu\,|q|^2 q = 0, \quad (x,t) \in \mathbb{R} \tag{4}$$

  [$\nu = -1$ and $\nu = 1$ $\Leftrightarrow$ "focusing" and "defocusing" NLS]
  has many applications in deep water waves, Bose-Einstein condensates, propagation of light in nonlinear optical fibers, etc.

- The NLS equation is an integrable system, which means that there exists an operator pair **X,T**, the so-called Lax pair such that **X,T** satisfy:

$$\Phi_x(x,t,k) = \mathbf{X}(x,t,k)\Phi(x,t,k), \quad \Phi_t(x,t,k) = \mathbf{T}(x,t,k)\Phi(x,t,k) \tag{5}$$

  if and only if the compatibility condition $\Phi_{xt} = \Phi_{tx}$ is identically satisfied provided q solves (4).

- For the focusing NLS equation:

$$\mathbf{X}(x,t,k) = \begin{pmatrix} -ik & q \\ -q^* & ik \end{pmatrix}, \quad \mathbf{T}(x,t,k) = \begin{pmatrix} -2ik^2 + i|q|^2 & 2kq + iq_x \\ -2kq^* + iq_x^* & 2ik^2 - i|q|^2 \end{pmatrix}$$

  where $k \in \mathbb{C}$ is the scattering parameter (similar to the Fourier spectral parameter). Φ are called the scattering eigenfunctions, and are functions of x, t, k.

IST: analog of a nonlinear Fourier transform

The existence of a Lax pair implies that we can solve the initial value problem of the NLS equation via the inverse scattering transform (IST), which consists of the following three steps:

- **direct scattering problem [analog of direct Fourier transform]:** the initial data $q(x,0)$ is transformed into the scattering data, denoted by $S(k,0)$: a reflection coefficient $\rho(k,0)$ [analog of the Fourier transform of the initial condition], discrete eigenvalues k_j's, which are the values of k where the problem admits bounded eigenfunctions [each k_j associated to one soliton], and norming constants, often denoted by C_j's, which specify the normalization of the eigenfunctions.

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- **time dependence problem** **[analog of time evolution in Fourier space]**: time evolution of the transformed data, i.e., determine $S(k, t)$ from $S(k, 0)$ via simple, explicitly solvable ODEs.
- **inverse scattering problem** **[analog of inverse Fourier transform]**: recovery of the evolved solution $q(x, t)$ from the evolved solution $S(k, t)$.

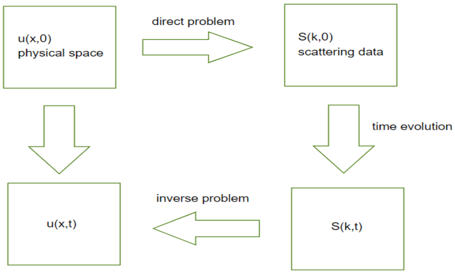


Figure: IST schematically

- Gardner, Greene, Kruskal & Miura in 1967 solved the initial value problem for the KdV equation using ideas of direct/inverse scattering.
- The technique was then generalized by Zakharov & Shabat in 1972 and used to integrate the NLS equation.
- Ablowitz, Kaup, Newell & Segur in 1974 showed that the technique applies to a wide class of nonlinear PDEs.

Some physically relevant integrable systems

- Integrable systems in $1+1$ dimensions
  - Korteweg-de Vries equation (KdV)
  - Nonlinear Schrödinger equation (NLS)
  - Modified KdV equation
  - Sine-Gordon equation
  - Derivative NLS equation
  - Benjamin-Ono equation
  - Boussinesq equation
  - Camassa-Holm equation
  - Classical Heisenberg ferromagnet model
  - Degasperis-Procesi equation
  - Harry-Dym equation
  - Kaup-Kupershmidt equation
  - Krichever-Novikov equation
  - Landau-Lifshitz equation
  - Nonlinear sigma models
  - Thirring model
  - Three-wave interaction equation
  - Maxwell-Bloch systems
  - Short-pulse equations
  - ...
- Integrable PDEs in $2+1$ dimensions
  - Kadomtsev-Petviashvili equation
  - Davey-Stewartson equation
  - Ishimori equation
  - Novikov-Veselov equation
  - ...
- Integrable lattice models
  - Ablowitz-Ladik lattice
  - Toda lattice
  - Volterra lattice
  - ...

Riemann-Hilbert Analysis

Reconstruction of the solution

There is a direct mapping between the initial data of the potential, $q(x,0)$, and the scattering data, $S(k,t)$, as established in the direct problem of the IST. Consequently, a one-to-one correspondence exists between the solution $q(x,t)$ of the NLS equation and a matrix-valued function $M(x,t;k)$ which satisfies a matrix Riemann Hilbert problem (RHP). In particular, $q(x,t)$ is obtained from $M(x,t;k)$ via the relation:

$$q(x,t) = -2i \lim_{k\to\infty} \left[k\, M_{12}(x,t;k)\right]. \tag{6}$$

RHP for M

Find 2×2 matrix-valued function $M(x,t;k)$ such that:

1. $M(x,t;k)$ is sectionally analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $M(x,t;k)$ has well-defined boundary values $M_\pm(x,t;k)$ on $\mathbb{R}$ that satisfy:

$$M_+(x,t;k) = M_-(x,t;k) \begin{pmatrix} 1+|\rho(k)|^2 & \rho^*(k)\, e^{2i\,\Omega(x,t;k)} \\ \rho(k)\, e^{-2i\,\Omega(x,t;k)} & 1 \end{pmatrix}, \quad k \in \mathbb{R} \tag{7}$$

$$\Omega(x,t;k) = kx - 2k^2 t$$

3. $M(x,t;k)$ has simple poles at $\{k_j, k_j^*\}_{j=1}^N$, with residues:

$$\operatorname{Res}_{k=k_j} M(x,t;k) = \lim_{k\to k_j} \left[M(x,t;k) \begin{pmatrix} 0 & 0 \\ C_j e^{2i\,\Omega(x,t;k_j)} & 0 \end{pmatrix} \right] \tag{8}$$

$$\operatorname{Res}_{k=k_j^*} M(x,t;k) = \lim_{k\to k_j^*} \left[M(x,t;k) \begin{pmatrix} 0 & -C_j^* e^{-2i\,\Omega(x,t;k_j^*)} \\ -0 & 0 \end{pmatrix} \right] \tag{9}$$

4. $M(x,t;k) = I_2 + O(k^{-1}), \quad k \to \infty.$

Solitonic solutions

To study pure solitonic solutions (no reflection) we set $\rho(k) = 0$ for all $k \in \mathbb{R}$. In this case, the previous RHP reduces to finding a meromorphic function $M(x, t; k)$ with:

1. poles at $\{k_j, k_j^*\}_{j=1}^N$ with the given residue conditions (N is the $\#$ of solitons)
2. no jump across $\mathbb{R}$
3. $M(x, t; k) = I_2 + O(k^{-1}), \quad k \to \infty.$

We can express:

$$M(x,t;k) = I_2 + \sum_{j=1}^{N} \frac{1}{k-k_j} \begin{pmatrix} \alpha_1^j & 0 \\ \alpha_2^j & 0 \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{k-k_j^*} \begin{pmatrix} 0 & \beta_1^j \\ 0 & \beta_2^j \end{pmatrix} \tag{10}$$

where α_ℓ^j and β_ℓ^j are constants independent of k, for $\ell = 1, 2$. Imposing the known residue conditions, we write a system for α_ℓ^j and β_ℓ^j which we can solve and compute $M(x, t; k)$ explicitly.

Remark

When $\rho(k)$ is present, then we cannot solve for $M(x, t; k)$ in a closed form but we can still express it in terms of algebraic-integral equations.

- Consider the defocusing NLS equation

$$iq_t + q_{xx} - 2|q|^2 q = 0 \tag{11}$$

with NZBCs

$$q(x,t) \to q_\pm(t), \quad x \to \pm\infty$$

then the system admits dark solitons (localized intensity dips over the nonzero background).

- The IST for NLS systems with NZBCs is much more challenging!

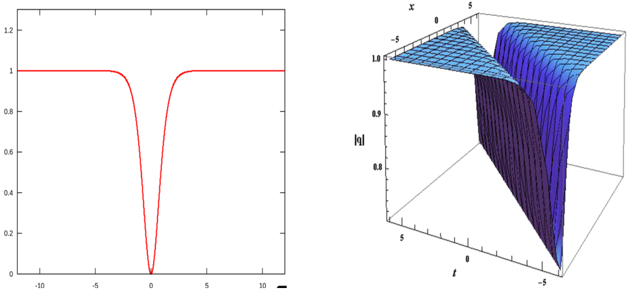


Figure: Dark soliton solutions.

Recent developments: a numerical IST

Problem setup

Consider the modified defocusing NLS equation

$$iq_t + q_{xx} + 2(q_o^2 - |q|^2)q = 0 \tag{12}$$

with constant NZBCs at infinity

$$q_\pm = \lim_{x\to\pm\infty} q(x,t) = q_o e^{i\theta_\pm}, \tag{13}$$

and piecewise constant initial condition (IC) of box-type

$$q(x,0) = \begin{cases} q_- = q_0 e^{-i\theta_-}, & x < -L \\ q_c = he^{i\alpha}, & -L < x < L \\ q_+ = q_0 e^{i\theta_+}, & x > L \end{cases} \tag{14}$$

where h, q_0 and L are arbitrary non-negative parameters, $\theta_\pm, \alpha$ are arbitrary phases (the term q_o^2 in the PDE makes the boundary conditions independent of time).

Our aim

- Our goal is to solve numerically equation (12) equipped with conditions (13) and (14), for the values

$$L = 1, \quad q_o = 1, \quad \theta = 0, \quad \alpha = 0, \quad h > q_o$$

  and study the time evolution of the box-type IC.

- Analytic solution is not possible (ρ is present). Numerics is needed.

- However, traditional direct numerical methods are difficult to handle discontinuous ICs $\Rightarrow$ Gibbs phenomena.

- We use a combination of analysis and numerics.

Reflection coefficient

For this piece-wise IC, we can derive the explicit expression of the reflection coefficient $\rho(z)$:

$$\rho(z) = e^{-2i\lambda(z)} \left(\frac{ik(z)(h-1)}{k^2(z) - h + i\lambda(z)\mu(z)\cot(2\mu(z))} \right). \tag{15}$$

Here, z is the so-called uniformization variable [Faddeev & Takthajan, Springer 1984] defined as:

$$z = k + \lambda \quad \Leftrightarrow \quad k = \frac{1}{2}(z + 1/z), \quad \lambda = \frac{1}{2}(z - 1/z) \tag{16}$$

and the function $\mu(z)$ is defined as: $\mu(z) = \sqrt{k^2(z) - h^2}$.

Advantages of choosing this discontinuous IC

- Explicit expression of the scattering data.
- The specific choice of IC yields analyticity of $\rho(z)$ in the entire z-plane.
- For this specific choice of IC, the problem admits no discrete eigenvalues$\Rightarrow$ no poles for ρ and M.

RHP for M

Find a 2×2 matrix-valued function $M(z) := M(x, t; z)$ such that:

1. $M(z)$ is sectionally analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $M(z)$ has well-defined boundary values $M_\pm(z)$ on $\mathbb{R} \setminus \{0\}$ that satisfy:

$$M_+(z) = M_-(z) G_M(x, t; z), \quad G_M(x, t; z) = \begin{pmatrix} 1 - |\rho(z)|^2 & -\rho^*(z) e^{2i\Omega(x,t;z)} \\ \rho(z) e^{-2i\Omega(x,t;z)} & 1 \end{pmatrix}$$

$$\Omega(x, t; z) = -\lambda(z)x - 2k(z)\lambda(z)t$$

3. $M(z)$ admits the asymptotic behavior:

$$M(z) = I_2 + \mathcal{O}(1/z), \quad z \to \infty \tag{18a}$$

$$M(z) = \frac{\sigma_2}{z} + \mathcal{O}(z), \quad z \to 0 \tag{18b}$$

where $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Our aim

Our goal is to find a 2×2 matrix-valued function M which satisfies conditions 1.-3. In other words, we want to solve numerically the RHP for M and obtain the solution to the defocusing NLS equation via the aforementioned relation

$$q(x,t) = \lim_{\substack{z \to \infty \\ z \in \mathbb{C}^+}} \left[iz\, M_{1,2}(x,t;z) \right]. \tag{19}$$

Numerical solution of RHPs connected to integrables PDEs over the years

- 2012: Trogdon, Olver & Deconinck - KdV and mKdV equations (ZBCs)
- 2013: Trogdon & Olver - focusing and defocusing NLS equation (ZBCs)
- 2017: Bilman & Trogdon - Toda lattice (ZBCs)
- 2020: Bilman & Trogdon - KdV equation (NZBCs)
- 2024: Gkogkou, Prinari & Trogdon - defocusing NLS equation (NZBCs)

Analysis: main ingredients

- Removing the singularity at $z = 0$ by introducing a new function that satisfies the same RHP as M and is well-defined at $z = 0$.
- Eliminate the rapidly oscillatory exponents appearing on the jump matrix

$$G_m(\xi; z) = \begin{pmatrix} 1 - |\rho(z)|^2 & -\rho^*(z)e^{-2t\, i\theta(\xi;z)} \\ \rho(z)e^{2t\, i\theta(\xi;z)} & 1 \end{pmatrix}$$

$$\theta(\xi; z) = 2\Big(\lambda(z)\xi + k(z)\lambda(z)\Big), \quad \xi = x/2t$$

and turn them into exponential decay, by performing contour deformations away from the real axis [opening lenses], following the principles of nonlinear steepest descent [analog to the method of steepest descent for approximating integrals](Deift & Zhou, 1993; Deift, Venakides & Zhou, 1997).

Removing the singularity at $z = 0$

Consider the function $\widetilde{M}$ which satisfies the following RHP.

RHP for $\widetilde{M}$

Find a 2×2 matrix-valued function $\widetilde{M}(z)$ such that:

1. $\widetilde{M}(z)$ is sectionally analytic in $\mathbb{C} \setminus \mathbb{R}$
2. $\widetilde{M}(z)$ satisfies the same jump relation across $\mathbb{R}$ as M
3. $\widetilde{M}(z) = I_2 + \mathcal{O}(1/z), \quad z \to \infty$
4. $\widetilde{M}(z)$ has well-defined boundary values as $z \to 0$ from $\mathbb{C}^{\pm}$ which are equal

$$\widetilde{M}(0) := \lim_{\substack{z \to 0 \\ z \in \mathbb{C}^{\pm}}} \widetilde{M}_{\pm}(z).$$

Recovery of M

We solve numerically the RHP for $\widetilde{M}$, and we recover M via the relation

$$M(z) = \left(I_2 + \frac{\sigma_2 q_o e^{-i\theta\sigma_3}}{z} \widetilde{M}^{-1}(0) \right) \widetilde{M}(z), \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{20}$$

One can show that $\det \widetilde{M}(x, t, z) = 1$, for all $z \in \mathbb{C}$.

Numerical solution of the RHP for $m_{2,d}(z)$

- There exists a sequence of transformations that deforms the contour away from $\mathbb{R}$, converting the rapidly oscillatory exponents into exponential decay.
- The final transformation in this sequence is denoted as $m_{2,d}$ which satisfies the jump relation:

$$m_{2,d+}(z) = m_{2,d-}(z)\, G_{m_{2,d}}(\xi; z), \quad z \in \Sigma' = \bigcup_{j=1}^{4} \Sigma_j. \tag{21}$$

- This RHP is solved numerically using the method of Olver, 2012; Trogdon & Olver, 2015, where Σ' is a union of line segments and $G_{m_{2,d}}$ satisfies some regularity conditions [by converting the RHP into a system of linear equations].

Reconstructing $q(x,t)$

We reconstruct the potential $q(x,t)$ in terms of $m_{2,d}$ via the relation

$$q(x,t) = i\left(\left(m_{2,d}^{(1)}(x,t)\right)_{12} + \left(\sigma_2 \Delta^{-1}(0) m_{2,d}^{-1}(x,t,0)\right)_{12}\right) \tag{22}$$

through unraveling the transformations

$$q(x,t) = \lim_{\substack{z\to\infty \\ z\in\mathbb{C}^+}} \left(iz\, m_{1,2}(x,t,z)\right), \quad M(z) \to \widetilde{M}(z) \to m_{1,d}(z) \to m_{2,d}(z). \tag{23}$$

Here, $m_{2,d}^{(1)}(x,t)$ is the $\mathcal{O}(1/z)$-order asymptotics of $m_{2,d}$ as $z \to \infty$.

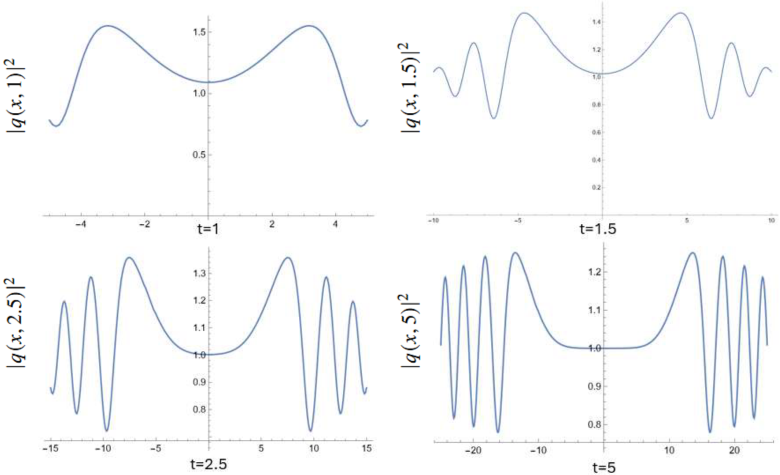


Figure: Plot of $|q(x,t)|^2$ for fixed values of t as a function of x (small spatial domain) for the chosen IC: $q_o = 1$, $L = 1$, $\theta_\pm = 0$, $\alpha = 0$ and $h = 1.5$.

Time evolution of the IC

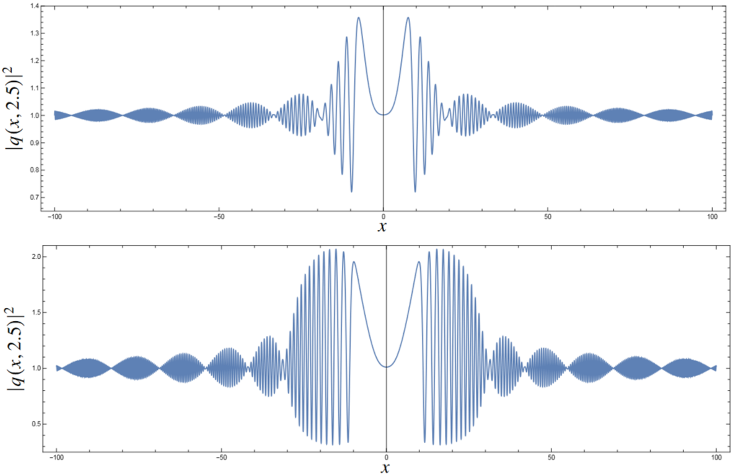


Figure: Plot of $|q(x,t)|^2$ for $t = 2.5$ as a function of x (larger spatial domain) for the values $q_o = 1,\ L = 1,\ \theta = 0,\ \alpha = 0, h = 1.5$ (top) and $q_o = 1,\ L = 1,\ \theta = 0,\ \alpha = 0, h = 3$ (bottom).

Recent developments: soliton gas

Figure: An original prop used in the production of the long-running BBC television series Doctor Who.

Soliton gas

- The concept of soliton gas was introduced in 1971 by Zakharov as an infinite collection of weakly interacting (KdV) solitons. In a diluted soliton gas, solitons with random parameters are almost non-overlapping.
- Central object: describe the distribution of solitons for spectral parameters and soliton centers that obey a suitable kinetic equation.
- This concept was recently extended to dense gases, solitons strongly and continuously interact.
- The notion of soliton gas is inherently associated with integrable PDEs.
- We are interested in the asymptotic behavior of the N-soliton solution of an integrable system when $N \to \infty$.
- Key tools: IST, Riemann-Hilbert Analysis, small-norm theory, thermodynamic limit of finite-gap potentials, generalized Gibbs ensembles.

Problem setup

- Consider the focusing NLS equation

$$iq_t + q_{xx} + 2|q|^2 q = 0. \tag{24}$$

- Set $\rho(k) = 0$, for all $k \in \mathbb{R}$ (no jump across $\mathbb{R}$).
- Choose N eigenvalues k_j accumulating on a horizontal line $\eta_1 \in \mathbb{C}^+$ (and k_j^* are accumulated in $\eta_2 \in \mathbb{C}^-$) and distributed via some density function.

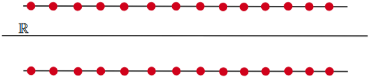


Figure: Poles k_j and their complex conjugates

- $q_N(x,t)$ is the corresponding N-soliton solution to the focusing NLS equation given by:

$$q_N(x,t) = \lim_{\substack{k\to\infty \\ z\in\mathbb{C}^+}} [iz\, M_{1,2}(x,t;k)] \tag{25}$$

where

$$M(x,t;k) = I_2 + \sum_{j=1}^{N} \frac{1}{k-k_j} \begin{pmatrix} \alpha_1^j & 0 \\ \alpha_2^j & 0 \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{k-k_j^*} \begin{pmatrix} 0 & \beta_1^j \\ 0 & \beta_2^j \end{pmatrix} \tag{26}$$

Our aim

Let $N \to \infty$ and study the asymptotic behavior of $q_N(x,t)$ when $N \to \infty$. In this case, $q_N(x,t)$ is approximated by the soliton gas condensate solution $q_{SG}(x,t,N)$, which is the corresponding solution to the NLS equation.

Theorem (G., Mazzuca, McLaughlin)

For all (x,t) in a compact set K, there is N_0 so that for all $N > N_0$, $|q_{SG}(x,t,N)|^2$ satisfies

$$|q_{SG}(x,t,N)|^2 = c^2 \frac{\left(\Theta'\left(\frac{\tau+1}{2};\tau\right)\right)^2}{\Theta(\tau;\tau)\Theta(0;\tau)} \frac{\Theta\left(-\frac{\zeta(x,N)}{2\pi}+\frac{1-\tau}{2};\tau\right)\Theta\left(-\frac{\zeta(x,N)}{2\pi}+\frac{\tau-1}{2};\tau\right)}{\Theta^2\left(-\frac{\zeta(x,N)}{2\pi};\tau\right)} + \mathcal{O}\left(\frac{1}{\log N}\right),$$

where the error term $\mathcal{O}\left(\frac{1}{\log N}\right)$ is uniform for all (x,t) in K. Here $\Theta(z;\tau)$ is the Jacobi Theta-3 function, and $\tau, c, \zeta(x,N), \Delta(x)$ are given by:

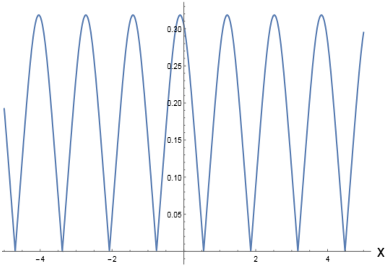
$$c = \frac{1}{2}\left(\int_A^{-A^*} \frac{1}{R_+(s)}ds\right)^{-1}, \quad \tau = 2c\int_{-A^*}^{-A} \frac{1}{R(s)}ds$$

$$\zeta(x,N) = -\frac{\tau\Delta(x) - 2\ln(N)}{2c\pi}\left(\int_{-A^*}^{-A} \frac{s^2}{R(s)}ds - \tau\int_A^{-A^*} \frac{s^2}{R_+(s)}ds\right)$$

$$\Delta(x) = \left(\int_{-A}^{-A^*} \frac{1}{R(s)}ds\right)^{-1}\left(-\int_A^{-A^*} \frac{\log\left(2\pi h(s)\rho(s)\right) + 2isx}{R_+(s)}ds + \int_{-A}^{A^*} \frac{\log\left(2\pi h(s)\rho(s)\right) - 2isx}{R_+(s)}ds\right)$$

where $R(z) = (z-A)^{\frac{1}{2}}(z+A)^{\frac{1}{2}}(z-A^*)^{\frac{1}{2}}(z+A^*)^{\frac{1}{2}}$ with standard branch-cuts, and A is the right endpoint of η_1.

Some plots



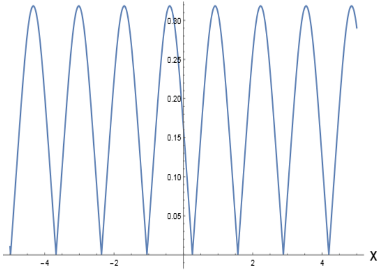


Figure: Solution to the focusing NLS equation. Here, $A = 1 + i$, $N = 2000$ on the left and $N = 1000$ on the right plot.

Remark

The leading order asymptotic behavior of $|q_{SG}(x, t, N)|^2$ is independent of time. This is because z_j accumulate on symmetric contours. $q_{SG}(x, t, N)$ can be interpreted as an elliptic wave with zero velocity [soliton gas condensate is in a sort of equilibrium state].

Thanks to my collaborators on the work I presented:

Kenneth McLaughlin, Tulane University

Barbara Prinari, University at Buffalo

Tom Trogdon, University of Washington

Guido Mazzuca, Tulane University

- Gkogkou A., Prinari B. & Trogdon T. *Numerical inverse scattering transform for the defocusing NLS with box-type initial conditions with nonzero background* (arXiv:2412.19703).
- Gkogkou A., Mazzuca G. & McLaughlin K. *The formation of a soliton gas condensate for the focusing Nonlinear Schrödinger equation* (arXiv:2502.14749).

Figure: Don't get eaten by a shark!

Thank you for your attention!

Any questions?

The classical IST: analysis

Recall that the focusing NLS equation:

$$iq_t + q_{xx} + 2|q|^2 q = 0, \quad (x,t) \in \mathbb{R} \tag{27}$$

admits the following Lax pair:

$$\Phi_x = \begin{pmatrix} -ik & q \\ -q^* & ik \end{pmatrix} \Phi, \qquad \Phi_t = \begin{pmatrix} -2ik^2 + i|q|^2 & 2kq + iq_x \\ -2kq^* + iq_x^* & 2ik^2 - i|q|^2 \end{pmatrix} \Phi. \tag{28}$$

Jost eigenfunctions

If $q \to 0$ (sufficiently rapidly) as $|x| \to \infty$, then the Lax pair asymptotically reduces to

$$\Phi_x \sim -ik\sigma_3\Phi, \qquad \Phi_t \sim -2ik^2\sigma_3\Phi, \qquad x \to \pm\infty \tag{29}$$

and one can define simultaneous solutions of the two equations of the Lax pair that satisfy:

$$\Phi_\pm(x,t;k) = (\Phi_{\pm,1}(x,t;k)\,, \Phi_{\pm,2}(x,t;k)) \sim I_2 e^{-i\Omega(x,t;k)\sigma_3}, \quad x \to \pm\infty\,,$$

where $\Omega(x,t;k) = kx - 2k^2 t$ and σ_3 is the third Pauli matrix.

If $q(.,t) \in L^1(\mathbb{R})$ for all $t \geq 0$, $\Phi_\pm(x,t;k)$ are continuous for all $k \in \mathbb{R}$, and $\Phi_{-,1}(x,t;k)$, $\Phi_{+,2}(x,t;k)$ are analytic for $k \in \mathbb{C}^+$, and $\Phi_{-,2}(x,t;k)$, $\Phi_{+,1}(x,t;k)$ are analytic for $k \in \mathbb{C}^-$.

Scattering coefficients

$\Phi_\pm$ are two fundamental matrix solutions of the Lax pair, therefore:

$$\Phi_-(x,t;k) = \Phi_+(x,t;k)\,S(k)\,, \quad S(k) = \begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{pmatrix}, \quad k \in \mathbb{R}$$

where

$$s_{11}(k) \equiv \mathrm{Wr}(\Phi_{-,1}\,, \Phi_{+,2}) \quad \text{analytic for } k \in \mathbb{C}^+$$
$$s_{22}(k) \equiv \mathrm{Wr}(\Phi_{+,1}\,, \Phi_{-,2}) \quad \text{analytic for } k \in \mathbb{C}^-$$

and reflection coefficient [analog of the Fourier transform of the initial condition]:

$$\rho(k) = \frac{s_{21}(k)}{s_{11}(k)}\,, \quad k \in \mathbb{R}\,.$$

The reflection coefficient plays a crucial role at the inverse problem of the IST.

Discrete eigenvalues

- Discrete eigenvalues are zeros of $s_{11}(k)$ for $k \in \mathbb{C}^+$ (resp., of $s_{22}(k)$ for $k \in \mathbb{C}^-$).
- By symmetry, discrete eigenvalues in the NLS equation appear in pairs: $\{k_n\,,\,k_n^*\}_{n=1}^{N}$, where N is the # of solitons.
- At each discrete eigenvalue, one has:

$$\mathrm{Res}_{k=k_n}[\Phi_{-,1}(x,t;k)/s_{11}(k)] = C_n\, e^{2i\Omega(x,t,k)}\, \Phi_{+,2}(x,t;k_n)\,,$$

C_n is the norming constant associated to the eigenvalue $k_n \in \mathbb{C}^+$, similarly for $k_n^* \in \mathbb{C}^-$.

Time evolution

- Because $\Phi_\pm(x,t;k)$ are defined as simultaneous solutions of the Lax pair, the evolution of the scattering data is trivial, namely, the scattering matrix $S(k)$ is independent of t. Consequently, the reflection coefficient, discrete eigenvalues, and normalization constants are all time-independent!
- One can work the two equations of the Lax pair separately. Then, the time evolution of the scattering data is given through simple differential equations.

Defocusing NLS: Lax pair and uniformization variable

The defocusing NLS equation

$$iq_t + q_{xx} - 2|q|^2 q = 0 \tag{30}$$

is an integrable system with Lax pair given by:

$$\mathbf{X}(x,t;k) = -ik\sigma_3 + Q, \quad \mathbf{T}(x,t;k) = -2ik^2\sigma_3 + i\sigma_3(Q_x - Q^2) + 2kQ, \tag{31a}$$

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ q^*(x,t) & 0 \end{pmatrix} \tag{31b}$$

such that **X,T** satisfy the linear problems

$$\Phi_x(x,t;k) = \mathbf{X}(x,t;k)\,\Phi(x,t;k), \quad \Phi_t(x,t;k) = \mathbf{T}(x,t;k)\,\Phi(x,t;k) \tag{32}$$

if and only if $\Phi_{xt} = \Phi_{tx}$ is identically satisfied provided q solves equation (30).

Defocusing NLS: Lax pair and uniformization variable

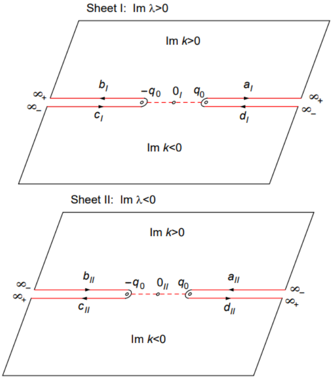
- As $x \to \pm\infty$, the scattering problem has eigenvalues: $\pm i\lambda$, with $\lambda = \sqrt{k^2 - q_o^2}$.
- Consequently, the continuous spectrum $\mathbb{R} \setminus (-q_o, q_o)$ has a gap, and the eigenfunctions have branching.
- To remove the branching of λ, one introduces a two-sheeted Riemann surface by gluing two copies of the complex k-plane cut along the semi-lines $(-\infty, -q_o] \cup [q_o, \infty)$.

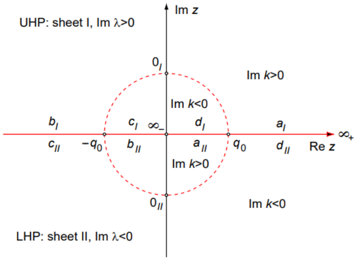
Defocusing NLS: Lax pair and uniformization variable

- Then define a uniformization variable [Faddeev & Takthajan, Springer 1984]:

$$z = k + \lambda \quad \Leftrightarrow \quad k = \frac{1}{2}(z + q_o^2/z)\,, \quad \lambda = \frac{1}{2}(z - q_o^2/z) \tag{33}$$

s.t. the branch cut on either sheet is mapped onto the real z axis, and sheet I/II is mapped onto the upper/lower half-plane of the z-plane:





A numerical IST: analysis

First deformation: opening lenses

Define the function $m_{1,d}(z) = \begin{cases} \widetilde{M}(z)P^{-1}(\xi,z), & z \in \Omega_1 \\ \widetilde{M}(z)U^{-1}(\xi,z), & z \in \Omega_2 \\ \widetilde{M}(z)L(\xi,z), & z \in \Omega_3 \\ \widetilde{M}(z)M(\xi,z), & z \in \Omega_4 \\ \widetilde{M}(z), & \text{elsewhere} \end{cases}$ which satisfies the jump condition:

$$m_{1,d_+}(z) = m_{1,d_-}(z) G_{m_{1,d}}(\xi,z), \quad G_{m_{1,d}}(\xi,z) = \begin{cases} P(\xi,z), & z \in \Sigma_1 \\ U(\xi,z), & z \in \Sigma_2 \\ L(\xi,z), & z \in \Sigma_3 \\ M(\xi,z), & z \in \Sigma_4 \\ D(z), & z \in (-\infty,0) \end{cases} \tag{34}$$

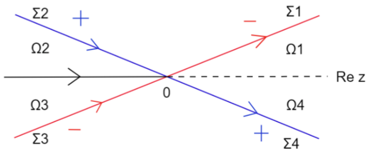


Figure: Opening lenses

where the matrices:

$$M(\xi,z)=\begin{pmatrix}1 & -\rho^*(z)e^{-2t\,i\theta(\xi,z)}\\ 0 & 1\end{pmatrix},\quad P(\xi,z)=\begin{pmatrix}1 & 0\\ \rho(z)e^{2t\,i\theta(\xi,z)} & 1\end{pmatrix} \tag{35a}$$

$$L(\xi,z)=\begin{pmatrix}1 & 0\\ \frac{\rho(z)}{1-|\rho(z)|^2}e^{2t\,i\theta(\xi,z)} & 1\end{pmatrix},\quad U(\xi,z)=\begin{pmatrix}1 & -\frac{\rho^*(z)}{1-|\rho(z)|^2}e^{-2t\,i\theta(\xi,z)}\\ 0 & 1\end{pmatrix} \tag{35b}$$

$$D(z)=\begin{pmatrix}1-|\rho(z)|^2 & 0\\ 0 & \frac{1}{1-|\rho(z)|^2}.\end{pmatrix} \tag{35c}$$

are such that

$$G_M(\xi,z)=M(\xi,z)P(\xi,z)\equiv L(\xi,z)D(z)U(\xi,z). \tag{36}$$

For large t, the jumps across Σ_j , for $j = 1, \cdots, 4$ are exponentially close to the identity matrix, as the exponents $e^{\pm 2ti\theta(\xi,z)}$ decay rapidly (because the segments Σ_j fall into regions where $Re(i\theta(\xi, z))$ has fixed and "desired" sign). This leads to increased efficiency in the numerical scheme, since fewer terms need to be computed. However, this does not happen with the jump matrix $D(z) = \begin{pmatrix} 1-|\rho(z)|^2 & 0 \\ 0 & \frac{1}{1-|\rho(z)|^2} \end{pmatrix}$ across $(-\infty, 0)$.

- We introduce the function $\Delta(z)$:

$$\Delta(z) = \begin{pmatrix} \delta_{11}(z) & 0 \\ 0 & 1/\delta_{11}(z) \end{pmatrix}, \quad \delta_{11}(z) = \exp\left(\frac{1}{2\pi i}\int_{-\infty}^{0} \frac{\log\left(1-|\rho(s)|^2\right)}{s-z}\, ds\right), \tag{37}$$

which satisfies:

$$\Delta_+(z) = \Delta_-(z)D(z), \quad z \in (-\infty, 0). \tag{38}$$

- Then, we define the function

$$m_{2,d}(z) = m_{1,d}(z)\,\Delta^{-1}(z). \tag{39}$$

It is then straightforward to check that $m_{2,d}$ is analytic on $(-\infty, 0)$.

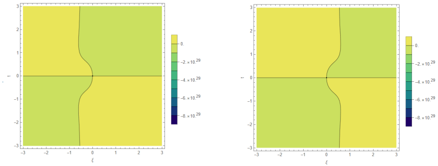


Figure: Left panel: sign chart of $Re(i\theta(\xi, z))$ when $0 < \xi < 1$. Right panel: sign chart of $Re(i\theta(\xi, z))$ when $-1 < \xi < 0$.

Second deformation: remove the jump $D(z)$

Then, the function $m_{2,d}$ satisfies the following RHP.

RHP for $m_{2,d}$: Find a 2×2 matrix-valued function $m_{2,d}$ such that:

1. $m_{2,d}(z)$ is analytic in $\mathbb{C} \setminus \Sigma$, where $\Sigma = \bigcup_{j=1}^{4} \Sigma_j$
2. $m_{2,d}(z)$ satisfies the jump relation across Σ

$$m_{2,d_+}(z) = m_{2,d_-}(z) G_{m_{2,d}}(\xi, z), \quad G_{m_{2,d}}(\xi, z) = \begin{cases} \Delta(z) P(\xi, z) \Delta^{-1}(z), & z \in \Sigma_1 \\ \Delta(z) U(\xi, z) \Delta^{-1}(z), & z \in \Sigma_2 \\ \Delta(z) L(\xi, z) \Delta^{-1}(z), & z \in \Sigma_3 \\ \Delta(z) M(\xi, z) \Delta^{-1}(z), & z \in \Sigma_4 \end{cases} \tag{40}$$

3. $m_{2,d}(z) = I_2 + \mathcal{O}(1/z), \quad z \to \infty$
4. The non-tangential limits of $m_{2,d}(z)$ as $z \to 0$ from $\mathbb{C}^{\pm}$ exist, and

$$m_{2,d+}(0) = m_{2,d-}(0) =: m_{2,d}(0).$$