Generalizing the Affirmative Action Problem: Mixing Numbers and Integrated Colorings of Graphs

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Motivation

- Proposed by Donald Newman

Concept

Consider a graph whose nodes are colored in one of two colors: black or white. A white node is called **integrated** if it has at least as many black neighbors as white neighbors, and similarly for a black node. A coloring of a graph is said to be integrated if each of the nodes are integrated.

Motivation

Question

Do all simple graphs admit an integrated coloring?

Short answer: YES!

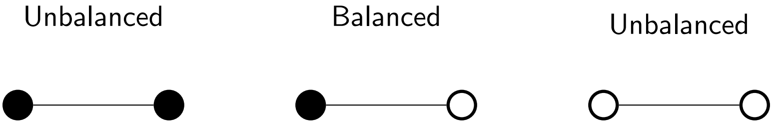
Definition

A **simple graph** is a graph $G = (V, E)$, where V is the set of vertices in G and E is the set of edges in G, such that G does not contain loops or multiple edges between any two vertices

Integrated Coloring

Definition

An edge is considered **balanced** if it is incident to both black and white vertices



Integrated Coloring

Theorem

If a graph G is colored in such a way it yields the maximum number of balanced edges, then G is integrated.

Proof.

Assume there is a coloring of some graph G with the maximum number of balanced edges and is not integrated. WLOG, there is a white node that is adjacent to more white vertices than black vertices. If we switch the color of this vertex from white to black, we increase the number of balanced edges. This contradicts the assumption that the coloring contains the maximum number of balanced edges. □

Mixing Number

Definition

For each vertex $v \in V$, the **mixing number** of v, or $\boldsymbol{mix(v)}$, is the number of opposite colored vertices adjacent to v. In other words, the total number of balanced edges incident to v.

Theorem

If a graph is integrated, then for all $v \in V$, $mix(v) \geq \frac{1}{2}deg(v)$

Mixing Number

Lemma

For every graph G,

$$\sum_{v \in V} mix(v) = 2(mix(G))$$

Mixing Number

Proof.

Using previous Lemma we have,

$$\begin{aligned} 2(mix(G)) &= \sum_{v \in V} mix(v) \\ &\geq \sum_{v \in V} \frac{1}{2} deg(v) \\ &= \frac{1}{2} \cdot 2\,|E| \\ &= |E| \end{aligned}$$

□

Converse of Affirmative Action Problem

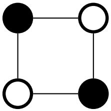
Question

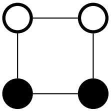
If a coloring of G is integrated, then does it contain the maximum possible number of balanced edges?

Answer: No!

Converse of Affirmative Action Problem

- Consider square circuit graphs:





Integrated Mixing Spectrum

Definition

The **integrated mixing spectrum** of a graph G, or ***ims*(*G*)**, is the set of all possible mixing numbers across all integrated colorings

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- Let $ims^-(G)$ and $ims^+(G)$ be the minimum and maximum, respectively, of $ims(G)$.

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Max Cut Problem

Question

How do you bound $ims(G)$?

- Answer: Using the Max Cut Problem

Max Cut Problem

Concept

Given a graph G, partition, or cut, V into two sets S and $V - S$ that has most possible edges connecting S to $V - S$

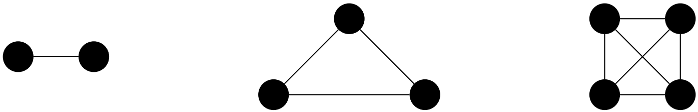
- WLOG, consider the sets S and $V - S$ to be the set of black and white vertices, respectively.
- By the previous proof, we know $mix(v) \geq \frac{1}{2}deg(v)$ for all $v \in V$, so $ims^{-}(G) \geq \frac{1}{2}|E|$
- Max cut $= ims^{+}(G)$

Complete Graphs

Definition

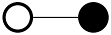
A complete graph, K_n, is a graph with n vertices that has every vertex adjacent to every other vertex.

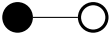
Examples



K_2

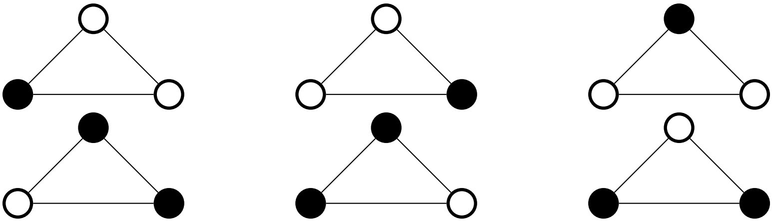
The number of integrated colorings on K_2 is trivial. We start with coloring one node black or white and then make the other node the unused color. We can do this twice:





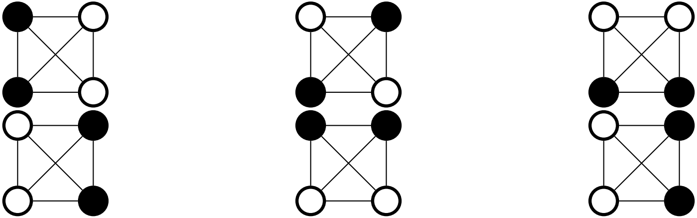
K_3

Finding $ic(K_3)$ is a little more complicated, but it is still easy to find all 6 possible integrated colorings.



K_4

K_4, despite having more vertices than K_3, has the same number of integrated colorings.



K_n?

- Is there a way to count the number of integrated colorings a complete graph has for any K_n?

K_n?

- Is there a way to count the number of integrated colorings a complete graph has for any K_n?
- Yes, but we have to consider the parity of the number of vertices in K_n.

Complete Graph Theorems

Theorem: $ic(K_{2n})$

The total number of integrated colorings on K_{2n} is $\binom{2n}{n}$

Theorem: $ic(K_{2n-1})$

The total number of integrated colorings on K_{2n-1} is $2\binom{2n-1}{n}$

Proof for K_{2n}

Let x be the number of black vertices in K_{2n} and y be the number of white vertices in K_{2n}

Then $x + y = 2n$. Suppose $x \neq y$. Without loss of generality, say $x > n$ so that then $y < n$. This means a black node would have at least n black nodes adjacent to it and no more than $n - 1$ white nodes adjacent to it. This would mean K_{2n} is not an integrated coloring so we need $x = y = n$.
From the $2n$ vertices, we make n of them black and the rest white. There is $\binom{2n}{n}$ ways to pick n vertices from $2n$. Since the remaining vertices are white, there $\binom{2n}{n}$ integrated colorings over K_{2n}.

Proof K_{2n-1}

Let x be the number of black nodes in K_{2n-1} and y be the number of white nodes in K_{2n-1}

Then $x + y = 2n - 1$. So one color has 1 more node than the other. Without loss of generality, suppose $y = x + 1$ so that then $x + y = 2x + 1 = 2n - 1$.
Then $2x = 2n - 2$ and so $x = n - 1$ and $y = n$.
So out of the $2n - 1$ vertices, we need n vertices 1 color and the other $n - 1$ vertices the other color. There are $\binom{2n-1}{n}$ ways of picking the n vertices from $n - 1$. However, there are two options for the color with n vertices so we have $2\binom{2n-1}{n}$ integrated colorings over K_{2n-1}.

Integrated Mixing Spectrum on K_n

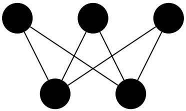
- $ims^+(K_{2n}) = ims^-(K_{2n}) = n^2$. Think number of choices for endpoints.
- $ims^+(K_{2n-1}) = ims^-(K_{2n-1}) = n(n-1)$. Again, number of choices for endpoints.

Complete Bipartite Graphs

Definition

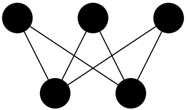
A complete bipartite graph, denoted by $K_{m,n}$, is defined as the graph whose set of vertices can be partitioned into two disjoint sets, L and R, where $|L| = m$ and $|R| = n$, and an edge (u, v) exists in the graph if and only if $u \in L$ and $v \in R$, or vise versa.

Example: $K_{3,2}$



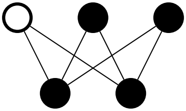
Motivating Example

How many integrated colorings can we form from the complete bipartite graph $K_{3,2}$? Let us balance the vertices one by one until all of the vertices are balanced.



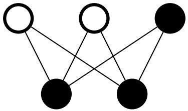
Integrating $K_{3,2}$

Vertex 1 would not be considered integrated if it remains the color black, as it is incident to two other black vertices. To satisfy the definition of integration, we must change the color of vertex 1 to white, thus creating a white vertex incident to two black vertices.



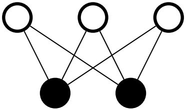
Integrating $K_{3,2}$ cont.

We will now consider vertex 2. Like vertex 1, it is a black vertex incident to 2 other black vertices, which means that it is not integrated. To achieve integration of vertex 2, we must change vertex 2's color to white. This is acceptable as vertex 2 remains incident only to the 2 black vertices.



Integrating $K_{3,2}$ cont.

We apply the same logic to vertex 3 and change the coloring of vertex 3 from black to white to ensure integration.

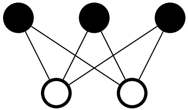


Motivating Question

We observe that all of the vertices in each disjoint set are of the same coloring in this configuration. Given this observation, is there another way to color $K_{3,2}$ that still satisfies the properties of integrated coloring?

Motivating Question

The anti-coloring of an integrated coloring, where the colors of all of the vertices are swapped, is still integrated. So, switching the colors of the top and bottom sets will still yield an integrated coloring.

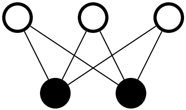


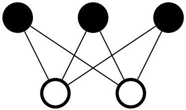
Notice: If we change the color of even one vertex so that it no longer matches the color of the other vertices in its set, whether on the top or the bottom, the coloring will no longer be integrated.

Integrated Colorings for $K_{3,2}$

There are two possible integrated colorings for $K_{3,2}$.

$$ic(K_{3,2}) = 2$$





Integrated Mixing Spectrum of $K_{3,2}$

The integrated mixing spectrum for $K_{3,2}$, is

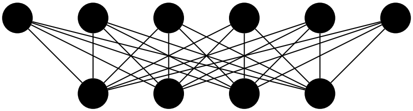
$$ims(K_{3,2}) = \{6\}$$

because both integrated colorings of $K_{3,2}$ yield a configuration in which all of the edges are balanced.

Another Motivating Example

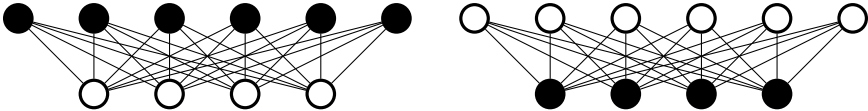
Now we will work with a complete bipartite graph with 2 disjoint sets of even cardinality instead of one even and one odd and observe the changes to the number of integrated colorings and the integrated mixing spectrum.

Example: $K_{6,4}$



Integrated Colorings of $K_{6,4}$

Let us begin with the simplest coloring. We will apply the same logic we derived from the previous complete bipartite graph coloring. Here are two possible integrated colorings, all the vertices at the top are one color (white or black) and the bottom vertices are colored in the opposite color.



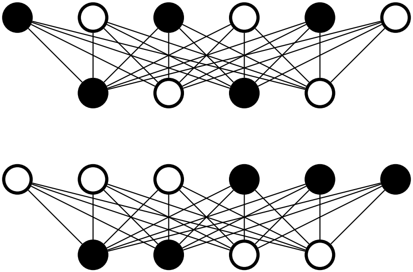
Integrated Colorings of $K_{6,4}$ cont.

The second possible option for integrated coloring is to assign colors to vertices such that the coloring is evenly split, exactly half of them black, and exactly half are white. And this is done for the top and the bottom vertices.

This distribution ensures that each of the vertices are incident to a sufficient number of vertices of the opposite color to satisfy integration properties.

Integrated Colorings of $K_{6,4}$ cont.

Some examples of that coloring



Note: There are other possible ways to split the colorings of the vertices

Motivating Question

Calculate the number of possible integrated colorings

To calculate the number of colorings when exactly half of the top and bottom vertices are a different color, we use combinations.

For the top vertices:

$$\binom{6}{3}.$$

For the bottom vertices:

$$\binom{4}{2}.$$

Thus

$$ic(K_{6,4}) = 2 + \binom{6}{3}\binom{4}{2}.$$

Integrated Mixing Spectrum of $K_{6,4}$

The integrated mixing spectrum is,

$$ims(K_{6,4}) = \{24, 12\}.$$

Where the 24 corresponds to Option 1, where the top and bottom vertices are opposite colors, and the 12 corresponds to Option 2, where each half of the top and bottom vertices are evenly split between the two colors.

Formula for Integrated Colorings and Integrating Mixing Spectrum of Complete Bipartite Graphs

We want to consider the complete bipartite graph $K_{m,n}$, where L and R is the disjoint bipartition of V. Let r represent the number vertices in L colored black, and so $m - r$ represents the number of vertices in L colored white.

Assume, WLOG, $r \geq m - r$

- If $r \neq m - r$ and $0 < r < m$, there there is an imbalance:
  - Vertices in the other disjoint set would be connected to more black than white vertices, so they will fail the condition for integration.
  - All of the vertices in the set n would need to be colored white.

Formula for Integrated Colorings and Integrating Mixing Spectrum of Complete Bipartite Graphs cont.

- If any of the vertices in R are white, then the white vertices in L will have all white neighbors, thus violating the condition for integration.

Conclusion: To have an integrated coloring in $K_{m,n}$, one of the following must be true

- $r = 0$
- $m = r$
- $r = m - r \Rightarrow 2r = m$

We can create a piecewise formula for $ic(K_{m,n})$

$$ic(K_{m,n}) = \begin{cases} 2 & \text{if either } m \text{ or } n \text{ is odd} \\ 2 + \binom{m}{m/2}\binom{n}{n/2} & \text{if } m,n \text{ are both even} \end{cases}$$

We can also create a piecewise function for $ims(K_{m,n})$

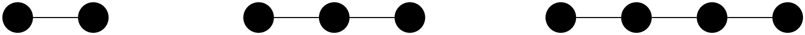
$$ims(K_{m,n}) = \begin{cases} \{mn\} & \text{if either } m \text{ or } n \text{ is odd} \\ \{mn/2, mn\} & \text{if } m,n \text{ are both even} \end{cases}$$

Path Graphs

Definition

The path graph, P_n, is a graph with n vertices connected in a single line by $n - 1$ edges.

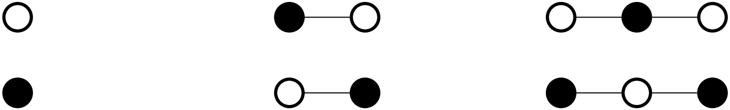
Examples



Integrated Colorings of P_n

For P_n to be integrated, you must avoid any substring of 3 consecutive White/Black nodes.

- This can also be thought of as avoiding 2 consecutive Unbalanced Edges (U).



P_1 P_2 P_3

$ic(P_n)$ Recurrence relation

Let $ic(P_n)$ denote the number of integrated colorings of P_n. It turns out the $ic(P_n)$ satisfies the recurrence relation for $n \geq 4$:

$$\boxed{ic(P_n) = ic(P_{n-1}) + ic(P_{n-2})}$$

Base Cases:

- $ic(P_1) = 2$
- $ic(P_2) = 2$
- $ic(P_3) = 2$

Fibonacci Connection

The integrated Coloring of path graphs follow a Fibonacci like recurrence.

$$\boxed{ic(P_n) = 2F_{n-1}}$$

- where F_n is the Fibonacci Sequence and $n \geq 2$.

n	$ic(P_n)$	F_n	$2F_{n-1}$
4	4	3	4
5	6	5	6
6	10	8	10
7	16	13	16
8	26	21	26

$ims(P_n)$

Every integrated coloring of P_n, must have a mixing number of at least $\lceil \frac{n-1}{2} \rceil$ and at most $n-1$.

Therefore,

$$ims^-(P_n) = \lceil \tfrac{n-1}{2} \rceil \qquad ims^+(P_n) = n-1$$

$$\boxed{ims(P_n) = \{k \in \mathbb{Z} : \lceil \tfrac{n-1}{2} \rceil \leq k \leq n-1\}}$$

Probability Distribution of $mix(C)$

Theorem

Let C be a random integrated coloring of P_n, and k be an integer between $\lceil \frac{n-1}{2} \rceil$ and $n-1$. Then, $Pr(mix(C) = k) = \frac{2\binom{k-1}{n-k-1}}{ic(P_n)}$

Proof.

Model C as a string of length of $n-1$ consisting of k Bs and $n-1-k$ Us with the following constraints:

- cannot have 2 consecutive Us;
- must begin and end with B.

This amounts to counting the number of distributions of $n-k$ nonempty blocks of Bs separated by the Us. A "stars and bars" argument shows there $\binom{k-1}{n-k-1}$ such strings. □

Probability Distribution pt.2

Theorem

Let C be a random integrated coloring of P_n, and k be an integer between $\lceil \frac{n-1}{2} \rceil$ and $n-1$. Then, $Pr(mix(C) = k) = \frac{2\binom{k-1}{n-k-1}}{ic(P_n)}$

Corollary

$$\sum_{k=\lceil \frac{n-1}{2} \rceil}^{n-1} 2\binom{k-1}{n-k-1} = ic(P_n)$$

This agrees with the well-known identity:

$$\sum_{k=\lceil \frac{n-1}{2} \rceil}^{n-1} \binom{k-1}{n-k-1} = F_{n-1}.$$

Generating Function

The generating function, $F(u,z) = \sum_{n=1}^{\infty} \sum_{C \in IC(P_n)} u^{mix(C)} z^n$, where

$IC(G)$ denotes the set of all integrated colorings of G,

satisfies the equation:

$$F(u,z) = 2z + uzF(u,z) + uz^2(F(u,z) - 2z)$$

Solving for $F(u,z)$ yields the bi-variate rational generating function

$$F(u,z) = \frac{2z - 2uz^3}{1 - uz - uz^2}$$

Cycle Graphs

Definition

The cycle graph, C_n, is the path graph P_n with an additional edge connecting vertex 1 and vertex n.

The integrated colorings of C_n can be modeled in the same way as they were for P_n, except now the strings

- cannot both begin and end with U (due to cyclicity);
- need an even number of Bs (otherwise the ending color of the cycle would not match the starting color).

Here, the recurrence relation for $ic(C_n)$ is, for $n \geq 7$,

$$ic(C_n) = ic(C_{n-2}) + 2ic(C_{n-3}) + ic(C_{n-4}),$$

with $ic(C_3) = 2$, $ic(C_4) = ic(C_5) = 6$, and $ic(C_6) = 10$.

$ims(C_n)$

Every integrated coloring of C_n, must have an even mixing number of at least $2\lceil \frac{n}{4} \rceil$ and at most $2\lfloor \frac{n}{2} \rfloor$.

Therefore,

$$ims^{-}(C_n) = 2\left\lceil \frac{n}{4} \right\rceil \qquad ims^{+}(C_n) = 2\left\lfloor \frac{n}{2} \right\rfloor$$

$$\boxed{ims(C_n) = \{2k : k \in \mathbb{Z}, \left\lceil \frac{n}{4} \right\rceil \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor\}}$$

Probability Distribution

Theorem

Let C be a random integrated coloring of C_n and k be an integer between $\lceil \frac{n}{4} \rceil$ and $\lfloor \frac{n}{2} \rfloor$. Then, $Pr(mix(C) = 2k) = \frac{2\binom{2k-1}{n-2k} + 4\binom{2k-1}{n-2k-1}}{\text{ic}(C_n)}$

Proof.

Model C as before with the aforementioned constraints. Then the string can either be bookended by B or begin and end with different letters. In the first case, the string comprises of $n - 2k + 1$ nonempty runs of Bs separated by the Us. A standard "stars and bars" argument shows that there are $\binom{2k-1}{n-2k}$ such strings. In the second case, the string comprises of $n - 2k$ nonempty runs of Bs separated by the Us. There are $\binom{2k-1}{n-2k-1}$ such strings. □

Generating Function

The generating function, $F(u,z) = \sum_{n=1}^{\infty} \sum_{C \in IC(C_n)} u^{mix(C)} z^n$

satisfies the equation: $F(u,z) =$
$2u^2z^2 + 6u^2z^3 + 4u^2z^4 + u^2z^2F(u,z) + 2u^2z^2F(u,z) + u^2z^4F(u,z)$.

Solving for $F(u,z)$ yields the bi-variate rational generating function

$$F(u,z) = \frac{2u^2z^2(2z^2+3z+1)}{1-u^2z^2(2z+1)}.$$

Limiting Distribution

- The rational generating functions for $ic(P_n)$ and $ic(C_n)$ suggest that $mix(C)$ satisfies a Central Limit Theorem for both classes of graphs.
- In particular, if μ_n and σ_n are the associated expected values and standard deviations of $mix(C)$ over uniform random path/cycle graphs of order n, then

$$\lim_{n \to \infty} \Pr\left[\frac{mix(C) - \mu_n}{\sigma_n} \leq x\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt.$$

- Asymptotic expansions of μ_n and σ_n, along with the convergence rate of the above limit law, can be ascertained by singularity analysis on the generating functions.